Binary Stars

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[For Examples class 1.30–3.30pm, Tuesday 26th April] Lent Term, 2022

Examples Sheet III

1. For the case of non-dissipative tides in a misaligned binary system show that

$$\dot{\mathbf{h}} = -\frac{M_2 B}{2\mu ablh^2}\mathbf{\Omega}.\mathbf{h}\,\mathbf{\Omega}\times\mathbf{h}$$

when averaged over an orbit. Deduce that both $\mathbf{\Omega}$ and $\mathbf{h}$ precess about the total angular momentum vector $\mathbf{H} = \mu\mathbf{h} + I\mathbf{\Omega}$ and find the time to precess once around.

2. From the tidal perturbative force

$$\mathbf{f} = -\frac{9M_2^2}{M_1 R^2(1-Q)^2\tau_{\rm damp}}\left(\frac{R_1}{r}\right)^{10}\left\{2\mathbf{r}(\mathbf{r}.\dot{\mathbf{r}}) + r^2(\dot{\mathbf{r}} - \mathbf{\Omega}\times\mathbf{r})\right\}$$

confirm that, when $\mathbf{\Omega}$ is parallel to $\mathbf{h}$ and changes are on a timescale much longer than the orbital period,

$$\dot{\mathbf{h}} = -\frac{2}{\tau_{\rm tid}}\left\{\frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1-e^2)^{13/2}} - \frac{\Omega}{\omega}\frac{1+3e^2+\frac{3}{8}e^4}{(1-e^2)^5}\right\}\mathbf{h},$$

where

$$\tau_{\rm tid} = \frac{2}{9}\tau_{\rm damp}\left(\frac{a}{R_1}\right)^8\frac{M_1^2}{M_2 M}(1-Q)^2$$

and a, e and ω are the semi-major axis, eccentricity and mean angular velocity of the orbit. Show further that

$$\frac{2\dot{\omega}}{3\omega} = -\frac{\dot{a}}{a} = \frac{\dot{E}}{E} = \frac{2}{\tau_{\rm tid}}\left\{\frac{1+\frac{31}{2}e^2+\frac{255}{8}e^4+\frac{185}{16}e^6+\frac{25}{64}e^8}{(1-e^2)^{15/2}} - \frac{\Omega}{\omega}\frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1-e^2)^6}\right\}.$$

Use conservation of angular momentum between stellar spin and orbit to confirm that

$$\frac{\dot{\Omega}}{\Omega} = -\frac{\mu h}{I\Omega}\frac{\dot{h}}{h},$$

where I is the moment of inertia of the expanded star spinning at Ω and μh is the moment of inertia of the orbit of mean angular velocity ω. Thence deduce that when $\mu h \gg I$ the spin of the star pseudosynchronises with the orbit at periastron so that

$$\Omega = \frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1+3e^2+\frac{3}{8}e^4)(1-e^2)^{3/2}}\omega$$

and sketch the ratio of this to the orbital angular velocity at periastron as a function of e.

3*. A star, of radius R, is tidally influenced by a point mass companion at a separation $\mathbf{a}$ with $a \gg R$. This generates a fluid flow $\mathbf{v}(\mathbf{r})$ at a position $\mathbf{r}$ relative to the centre of the perturbed star that is governed by continuity

$$\frac{\partial \rho}{\partial t} + \nabla.\rho\mathbf{v} = 0,$$

where $\rho(\mathbf{r})$ is the density which must be constant on equipotential surfaces described by

$$\bar{r} = r[1 + \alpha(r)P_2(\cos\theta)].$$

The function $\alpha(r)$ is the solution to Clairaut's equation for an unperturbed star and θ is the angle between $\mathbf{a}$ and $\mathbf{r}$ which lies on the equipotential surface for which $\bar{r} = \text{const}$. Show that the velocity field $\mathbf{v}$ can be written, to first order in α, as

$$\mathbf{v} = -\frac{1}{2}\beta(r)\alpha(R)\nabla K,$$

where $\beta(r)$ depends only on the structure of the unperturbed star and K is the harmonic function

$$K = \frac{\partial}{\partial t}\left\{\frac{3}{2}r^5 a^3 l_{ij}(\mathbf{r}) l_{ij}(\mathbf{a})\right\}.$$

Deduce that

$$\beta(r) = \frac{1}{\rho(r)}\int_R^r \frac{\alpha(r')}{\alpha(R)}\frac{d\rho}{dr'}\,dr',$$

in which the limits are chosen to ensure that β is not singular at the surface where $\rho \to 0$.

Thence show that

$$v_i = \frac{1}{QM_1R^2}\frac{\partial q_{ij}}{\partial t} r_j \beta(r),$$

where q_{ij} is the quadrupole tensor induced by the companion.

The rate of strain tensor

$$t_{ij} = (\nabla\mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}.$$

By taking the average of the square of this over equipotential surfaces, which may now be assumed to be spherical, show that the rate of dissipation of energy is

$$\begin{aligned} -\frac{dE}{dt} &= \frac{1}{2}\int_V \rho\nu t_{ij}^2\,dV \\ &= \frac{9M_2^2B^2}{M_1^2R^4Q^2}\frac{1}{a^{10}}\frac{\partial \mathbf{a}}{\partial t}\left\{2\mathbf{a}\left(\mathbf{a}.\frac{\partial \mathbf{a}}{\partial t}\right) + a^2\frac{\partial \mathbf{a}}{\partial t}\right\}\int_0^{M_1}\nu\gamma(r)\,dm, \end{aligned}$$

where

$$\gamma = \beta^2 + \frac{2}{3} r \beta \beta' + \frac{7}{30} \beta'^2,$$

ν is the kinematic viscosity in the fluid and Q and B are properties of the unperturbed star as defined in the lectures.

Thence show that the dissipation constant σ defined in the lectures is related to the the tidal lag time τ and the viscous damping timescale defined by,

$$\frac{1}{t_{\mathrm{damp}}} = \frac{1}{M_1 R^2} \int_0^{M_1} \nu \gamma \, dm,$$

by

$$\sigma = \frac{G\tau}{3B} = \frac{2}{M_1 R^2 Q^2} \frac{1}{t_{\mathrm{damp}}}.$$

4. In an eccentric binary system star 2 is accreting from the fast wind of star 1. Recall that the accretion rate averaged over an orbit is

$$\left\langle \dot{M}_2 \right\rangle = -\frac{\dot{M}_1 (GM_2)^2}{v_{\mathrm{w}}^4 a^2 \sqrt{1 - e^2}},$$

where the wind speed $v_{\mathrm{w}} \gg v_{\mathrm{orb}}$, the relative orbital speed of the stars. By considering only the momentum gained by star 2 in the material it accretes from the wind show that Newton's law for the system becomes

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3} \mathbf{r} + \frac{\dot{M}_2}{M_2} v_{\mathrm{w}} \frac{\mathbf{r}}{r}.$$

Use this force to evaluate the changes in specific energy, angular momentum and the Laplace-Runge-Lenz vector and hence the rate of change of separation a and eccentricity e when the effects are averaged over an orbit.

5. The red star in a cataclysmic variable has a surface magnetic field B_*. Outside the star this falls off like a dipole so that

$$\frac{B}{B_*} = \left(\frac{R}{R_*} \right)^{-3}$$

at a distance R from the centre of the star which has radius R_*. Approximating the wind of this star as an equatorial outflow at constant speed v show that the Alfvén radius R_{A} is given by

$$\left(\frac{R_{\mathrm{A}}}{R_*} \right)^4 = \frac{4\pi B_*^2 R_*^2}{\dot{M} v \mu_0},$$

where $\dot{M}$ is the rate of mass loss in the stellar wind.

The radius of the donor star of mass M_2 behaves as

$$\frac{R_*}{R_\odot} = \left(\frac{M_2}{M_\odot}\right)^{\frac{13}{15}} \quad \text{and} \quad \frac{R_\mathrm{L}}{a} = 0.462\left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where a is the separation of the circular orbit and M is the total mass of the system. Suppose that magnetic braking is the only source of angular momentum loss and $-\dot{M} \ll -\dot{M}_2 \approx \dot{M}_1$ to show that the rate of steady mass transfer is

$$\dot{M}_2 = \frac{15q}{19-15q}\frac{M}{M_1}\left(\frac{R_\mathrm{A}}{a}\right)^2 \dot{M} \propto \dot{M}^{1/2},$$

where $q = M_2/M_1$.

For a system with $M_1 = 0.6\,M_\odot$ and $M_2 = 0.3\,M_\odot$ in which v is the escape velocity from star 2 and $B_* = 10^3\,\mathrm{G} = 0.1\,\mathrm{T}$, what mass-loss rate $\dot{M}$ is required to maintain a mass-transfer rate of $\dot{M}_2 = 10^{-10}\,M_\odot\,\mathrm{yr}^{-1}$? What is R_A/R_* in this case?

6. A supergiant of C/O core mass M_c and envelope mass M_env, of which the binding energy may be expressed as

$$E_\mathrm{bind} \approx -\frac{2GM_\mathrm{env}^2}{R_\mathrm{G}},$$

where R_G is a fiducial radius defined by

$$\frac{R_\mathrm{G}}{R_\odot} \approx 1{,}000\left(\frac{M_\mathrm{c}}{M_\odot}\right)^2\left(\frac{M_\mathrm{env}}{M_\odot}\right)^{-\frac{1}{3}},$$

is in a binary with a C/O white dwarf of mass M_wd.

The giant fills its Roche lobe and dynamical mass transfer leading to common-envelope evolution ensues. Show that, if the common envelope efficiency is α_ce, the final separation of the cores when the envelope has been lost is a_f where

$$\frac{a_\mathrm{f}}{R_\mathrm{G}} \approx \frac{\alpha_\mathrm{ce}}{2}\frac{M_\mathrm{c}M_\mathrm{wd}}{M_\mathrm{env}^2},$$

in the limit $a_\mathrm{f} \ll a_\mathrm{i}$, where a_i is the initial separation.

The radius of a white dwarf of mass M_wd can be approximated by

$$\frac{R_\mathrm{wd}}{R_\odot} \approx 0.01\left(\frac{M_\odot}{M_\mathrm{wd}}\right)^{\frac{1}{3}}$$

and the radius of the hot giant core by

$$R_{\rm c}\left(M_{\rm c}\right) \approx 5 R_{\rm wd}\left(M_{\rm c}\right)$$

The spiralling cores coalesce if $a_{\rm f} \leq 3 \max\left(R_{\rm wd}, R_{\rm c}\right)$. Estimate the minimum envelope mass $M_{\rm crit}$ required for the cores to coalesce if $M_{\rm c} = 0.6\, M_{\odot}$, $M_{\rm wd} = 0.9\, M_{\odot}$ and $\alpha_{\rm ce} = 1.0$.

Suppose the process of coalescence heats the degenerate white dwarf and the supergiant core to a temperature at which carbon burning ${}^{12}{\rm C} + {}^{12}{\rm C} \rightarrow {}^{24}{\rm Mg}$ (13.93 MeV per reaction) can ignite. Estimate the total nuclear energy that can be released and compare it with the binding energy of the white dwarf which may be modelled as an $n = 3/2$ polytrope.

Comment on the result for $M_{\rm env}$ in the range from well below to well above $M_{\rm crit}$.

7. A binary system consists of two stars of masses M_1 and M_2 in an orbit with eccentricity e and semi-major axis a. At a point on the orbit when the separation is r, star 1 loses a fraction $1 - f$ of the total mass of the system $M = M_1 + M_2$ in an isotropic supernova explosion over a time short compared with the orbital period. After the explosion the total mass is $M' = fM$. Show that the system is most easily unbound if the explosion takes place at periastron in which case it unbinds if

$$f < \frac{1+e}{2}$$

and hence that such a supernova expelling half the mass in a system originally in a circular orbit unbinds the system.