Example Sheet 2

1. *Validity of the ideal MHD equations*

(a) Ohm's Law for a medium of electrical conductivity σ is $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{E}$ is the electric field measured in the rest frame of the conductor. Show that, in the presence of a finite and uniform conductivity, the ideal induction equation is modified to

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta\nabla^2\mathbf{B}\,,$$

where $\eta = 1/(\mu_0\sigma)$ is the *magnetic diffusivity.* Hence argue that the effects of finite conductivity are small if the magnetic Reynolds number $Rm = LU/\eta$ is large, where L and U are characteristic scales of length and velocity for the fluid flow.[1]

(b) Three effects neglected in a non-relativistic theory of MHD are (i) the displacement current in Maxwell's equations (compared to the electric current), (ii) the bulk electrostatic force on the fluid (compared to the magnetic Lorentz force) and (iii) the electrostatic energy (compared to the magnetic energy). Verify the self-consistency of these approximations by using order-of-magnitude estimates or scaling relations to show that these three neglected effects are all smaller than those retained by a factor of order U^2/c^2. [You may wish to consult Jackson, *Classical Electrodynamics* (Wiley), or a similar book.]

2. *Equilibrium of a solar prominence*

A simple model for a prominence or filament in the solar atmosphere involves a two-dimensional magnetostatic equilibrium in the (x, z) plane with uniform gravity $\mathbf{g} = -g\,\mathbf{e}_z$. The gas is isothermal, with isothermal sound speed c_s. The density and magnetic field depend only on x and the field lines become straight as $|x| \to \infty$.

Show that the solution is of the form

$$B_z = B_0 \tanh(kx)\,,$$

where k is a constant to be determined. Sketch the field lines and find the density distribution.

[1] The magnetic diffusivity in a fully ionized plasma is on the order of $10^{13}(T/\mathrm{K})^{-3/2}\,\mathrm{cm}^2\,\mathrm{s}^{-1}$. Simple estimates imply that $Rm \gg 1$ for observable solar phenomena.

3. *Equilibrium of a magnetic star*

A star contains an axisymmetric and purely toroidal magnetic field $\mathbf{B} = B(r, z)\,\mathbf{e}_\phi$, where (r, ϕ, z) are cylindrical polar coordinates. Show that the equation of magnetostatic equilibrium can be written in the form

$$\mathbf{0} = -\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}p - \frac{B}{\mu_0 r}\boldsymbol{\nabla}(rB)\,.$$

Assuming that the equilibrium is barotropic such that $\boldsymbol{\nabla}p$ is everywhere parallel to $\boldsymbol{\nabla}\rho$, show that the magnetic field must be of the form

$$B = \frac{1}{r}f(r^2\rho)\,,$$

where f is an arbitrary function. Sketch the topology of the contour lines of $r^2\rho$ in a star and argue that a magnetic field of this form is confined to the interior.

4. *Helicity*

The magnetic helicity in a volume V is

$$H_\mathrm{m} = \int_V \mathbf{A}\cdot\mathbf{B}\,dV\,.$$

A *thin, untwisted magnetic flux tube* is a thin tubular structure consisting of the neighbourhood of a smooth curve C, such that the magnetic field is confined within the tube and is parallel to C.

(a) Consider a simple example of a single, closed, untwisted magnetic flux tube such that

$$\mathbf{B} = B(r, z)\,\mathbf{e}_\phi\,,$$

where (r, ϕ, z) are cylindrical polar coordinates and $B(r, z)$ is a positive function localized near $(r = a, z = 0)$. The tube is contained entirely within V. Show that the magnetic helicity of this field is uniquely defined and equal to zero.

(b) Use the fact that H_m is conserved in ideal MHD to argue that the magnetic helicity of any single, closed, untwisted and unknotted flux tube contained within V is also zero.

(c) Consider a situation in which V contains two such flux tubes T_1 and T_2. Let F_1 and F_2 be the magnetic fluxes associated with T_1 and T_2. By writing $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, etc., and assuming that the tubes are thin, show that

$$H_\mathrm{m} = \pm 2F_1F_2$$

if the tubes are simply interlinked, while $H_\mathrm{m} = 0$ if they are unlinked.

5. *Variational principles*

The magnetic energy in a volume V bounded by a surface S is

$$E_{\mathrm{m}} = \int_V \frac{B^2}{2\mu_0}\,dV\,.$$

(a) Making use of the representation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ of the magnetic field in terms of a magnetic vector potential, show that the magnetic field that minimizes E_{m}, subject to the tangential components of $\mathbf{A}$ being specified on S, is a potential field. Argue that this constraint corresponds to specifying the normal component of $\mathbf{B}$ on S.

(b) Making use of the representation $\mathbf{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta$ of the magnetic field in terms of Euler potentials, show that the magnetic field that minimizes E_{m}, subject to α and β being specified on S, is a force-free field. Argue that this constraint corresponds to specifying the normal component of $\mathbf{B}$ on S and also the way in which points on S are connected by magnetic field lines.

Answers to questions 1 and 2 may be submitted for marking.
Please send any comments and corrections to `gio10@cam.ac.uk`