Example Sheet 3

1. *Shock relations*

The Rankine–Hugoniot relations in the rest frame of a non-magnetic shock are

$$[\rho u_x]_1^2 = 0\,,$$

$$\left[\rho u_x^2 + p\right]_1^2 = 0\,,$$

$$\left[\rho u_x\left(\tfrac{1}{2}u_x^2 + h\right)\right]_1^2 = 0\,,$$

where $u_x > 0$ and $[Q]_1^2 = Q_2 - Q_1$ is the difference between the downstream and upstream values of any quantity Q. Show that the velocity, density and pressure ratios

$$U = \frac{u_2}{u_1}\,, \qquad D = \frac{\rho_2}{\rho_1}\,, \qquad P = \frac{p_2}{p_1}$$

across a shock in a perfect gas are given by

$$D = \frac{1}{U} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}\,, \qquad P = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{(\gamma+1)}\,,$$

where $\mathcal{M} = u_x/v_\mathrm{s}$ is the Mach number, and also that

$$\mathcal{M}_2^2 = \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{2\gamma\mathcal{M}_1^2 - (\gamma-1)}\,.$$

Show that the entropy change in passing through the shock is given by

$$\frac{[s]_1^2}{c_v} = \ln P - \gamma \ln\left[\frac{(\gamma+1)P + (\gamma-1)}{(\gamma-1)P + (\gamma+1)}\right]$$

and deduce that only compression shocks ($D > 1$, $P > 1$) are physically realizable.

2. *The Riemann problem*

A perfect gas flows in one dimension in the absence of boundaries, gravity and magnetic fields.

(i) Determine all possible smooth local solutions of the equations of one-dimensional gas dynamics that depend only on the variable $\xi = x/t$ for $t > 0$. Show that one such solution is a rarefaction wave in which $du/d\xi = 2/(\gamma+1)$. How do the adiabatic sound speed and specific entropy vary with ξ?

(ii) At $t = 0$ the gas is initialized with uniform density ρ_{L}, pressure p_{L} and velocity u_{L} in the region $x < 0$ and with uniform density ρ_{R}, pressure p_{R} and velocity u_{R} in the region $x > 0$. Explain why the subsequent flow is of the similarity form described in part (i). What constraints must be satisfied by the initial values if the subsequent evolution is to involve only two uniform states connected by a rarefaction wave? Give a non-trivial example of such a solution.

(iii) Explain why, for more general choices of the initial values, the solution cannot have the simple form described in part (ii), even if $u_{\mathrm{R}} > u_{\mathrm{L}}$. What other features will appear in the solution? (Detailed calculations are not required.)

3. *Nonlinear waves in incompressible MHD*

Show that the equations of ideal MHD, in the case of an incompressible fluid of uniform density ρ, can be written in the symmetrical form

$$\frac{\partial \mathbf{z}_\pm}{\partial t} + \mathbf{z}_\mp \cdot \boldsymbol{\nabla} \mathbf{z}_\pm = -\boldsymbol{\nabla}\psi \,,$$

$$\boldsymbol{\nabla} \cdot \mathbf{z}_\pm = 0 \,,$$

where

$$\mathbf{z}_\pm = \mathbf{u} \pm \mathbf{v}_{\mathrm{a}}$$

are the *Elsässer variables*, $\mathbf{v}_{\mathrm{a}} = (\mu_0\rho)^{-1/2}\mathbf{B}$ is the vector Alfvén velocity, and $\psi = \Phi + (\Pi/\rho)$ is a modified pressure.

Consider a static basic state in which the magnetic field is uniform and $\psi =$ constant. Write down the exact equations governing perturbations $(\mathbf{z}'_\pm, \psi')$ (i.e. without performing a linearization). Hence show that there are special solutions in which disturbances of arbitrary amplitude propagate along the magnetic field lines in one direction or other without change of form. How do these relate to the MHD wave modes of a compressible fluid? Why does the general argument for wave steepening not apply to these nonlinear simple waves?

4. *Bondi accretion*

Write down the equations of steady, spherical accretion of a perfect gas in an arbitrary gravitational potential $\Phi(r)$.

Accretion on to a black hole can be approximated within a Newtonian theory by using the *Paczyński–Wiita potential*

$$\Phi = -\frac{GM}{r - r_{\mathrm{h}}} \,,$$

where $r_{\mathrm{h}} = 2GM/c^2$ is the radius of the event horizon and c is the speed of light.

Show that the sonic radius r_s is related to r_h and the nominal accretion radius $r_\mathrm{a} = GM/2v_\mathrm{s0}^2$ (where v_s0 is the sound speed at infinity) by

$$2r_\mathrm{s}^2 - [(5-3\gamma)r_\mathrm{a} + 4r_\mathrm{h}]\,r_\mathrm{s} + 2r_\mathrm{h}^2 - 4(\gamma-1)r_\mathrm{a}r_\mathrm{h} = 0\,.$$

Argue that the accretion flow passes through a unique sonic point for any value of $\gamma > 1$. Assuming that $v_\mathrm{s0} \ll c$, find approximations for r_s in the cases (i) $\gamma < 5/3$, (ii) $\gamma = 5/3$ and (iii) $\gamma > 5/3$.

5. *Critical points of magnetized outflows*

The integrals of the equations of ideal MHD for a steady axisymmetric outflow are

$$\mathbf{u} = \frac{k\mathbf{B}}{\rho} + r\omega\,\mathbf{e}_\phi\,, \tag{1}$$

$$u_\phi - \frac{B_\phi}{\mu_0 k} = \frac{\ell}{r}\,, \tag{2}$$

$$s = s(\psi)\,, \tag{3}$$

$$\tfrac{1}{2}|\mathbf{u} - r\omega\,\mathbf{e}_\phi|^2 + \Phi - \tfrac{1}{2}r^2\omega^2 + h = \varepsilon'\,, \tag{4}$$

where $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\varepsilon'(\psi)$ are surface functions. Assume that the magnetic flux function $\psi(r,z)$ is known from a solution of the Grad–Shafranov equation, and let the cylindrical radius r be used as a parameter along each magnetic field line. Then the poloidal magnetic field $\mathbf{B}_\mathrm{p} = \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi$ is a known function of r on each field line. Assume further that the surface functions $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\varepsilon'(\psi)$ are known.

Show that equations (1)–(3) can then be used, in principle, and together with the equation of state, to determine the velocity $\mathbf{u}$ and the specific enthalpy h as functions of ρ and r on each field line. Deduce that equation (4) has the form

$$f(\rho, r) = \varepsilon' = \text{constant}$$

on each field line.

Show that

$$-\rho\frac{\partial f}{\partial \rho} = \frac{u_\mathrm{p}^4 - (v_\mathrm{s}^2 + v_\mathrm{a}^2)u_\mathrm{p}^2 + v_\mathrm{s}^2 v_\mathrm{ap}^2}{u_\mathrm{p}^2 - v_\mathrm{ap}^2}\,,$$

where v_s is the adiabatic sound speed, v_a is the (total) Alfvén speed and the subscript 'p' denotes the poloidal (meridional) component. Deduce that the flow has critical points where u_p equals the phase speed of axisymmetric fast or slow magnetoacoustic waves. What condition must be satisfied by $\partial f/\partial r$ for the flow to pass through these critical points?

Answers to questions 2 and 4 may be submitted for marking.
Please send any comments and corrections to `gio10@cam.ac.uk`