Example Sheet 4

1. *Radial oscillations of a star*

Show that purely radial (i.e. spherically symmetric) oscillations of a spherical star satisfy the Sturm–Liouville equation

$$\frac{d}{dr}\left[\frac{\gamma p}{r^2}\frac{d}{dr}(r^2\xi_r)\right]-\frac{4}{r}\frac{dp}{dr}\xi_r+\rho\omega^2\xi_r=0\,.$$

How should ξ_r behave near the centre of the star and near the surface $r=R$ at which $p=0$?

Show that the associated variational principle can be written in the equivalent forms

$$\begin{aligned}\omega^2\int_0^R\rho|\xi_r|^2\,r^2\,dr&=\int_0^R\left[\frac{\gamma p}{r^2}\left|\frac{d}{dr}(r^2\xi_r)\right|^2+4r\frac{dp}{dr}|\xi_r|^2\right]dr\\&=\int_0^R\left[\gamma p r^4\left|\frac{d}{dr}\left(\frac{\xi_r}{r}\right)\right|^2+(4-3\gamma)r\frac{dp}{dr}|\xi_r|^2\right]dr\,,\end{aligned}$$

where γ is assumed to be independent of r. Deduce that the star is unstable to purely radial perturbations if and only if $\gamma<4/3$. Why does it not follow from the first form of the variational principle that the star is unstable for all values of γ?

Can you reach the same conclusion using only the virial theorem?

2. *Waves in an isothermal atmosphere*

Show that linear waves of frequency ω and horizontal wavenumber k_h in a plane-parallel isothermal atmosphere satisfy the equation

$$\frac{d^2\xi_z}{dz^2}-\frac{1}{H}\frac{d\xi_z}{dz}+\frac{(\gamma-1)}{\gamma^2H^2}\xi_z+(\omega^2-N^2)\left(\frac{1}{v_\mathrm{s}^2}-\frac{k_\mathrm{h}^2}{\omega^2}\right)\xi_z=0\,,$$

where H is the isothermal scale-height, N is the Brunt–Väisälä frequency and v_s is the adiabatic sound speed.

Consider solutions of the vertically wavelike form

$$\xi_z\propto e^{z/2H}e^{ik_zz}\,,$$

where k_z is real, so that the wave energy density (proportional to $\rho|\boldsymbol{\xi}|^2$) is independent of z. Obtain the dispersion relation connecting ω and $\mathbf{k}$. Assuming that $N^2>0$, show that propagating waves exist in the limits of high and low frequencies, for which

$$\omega^2\approx v_\mathrm{s}^2k^2\quad\text{(acoustic waves)}\qquad\text{and}\qquad\omega^2\approx\frac{N^2k_\mathrm{h}^2}{k^2}\quad\text{(gravity waves)}$$

respectively. Show that the minimum frequency at which acoustic waves propagate is $v_s/2H$.

Explain why the linear approximation must break down above some height in the atmosphere.

3. *Magnetic buoyancy instabilities*

A perfect gas forms a static atmosphere in a uniform gravitational field $-g\,\mathbf{e}_z$, where (x, y, z) are Cartesian coordinates. A horizontal magnetic field $B(z)\,\mathbf{e}_y$ is also present.

Derive the linearized equations governing small displacements of the form

$$\mathrm{Re}\left[\boldsymbol{\xi}(z)\exp(-i\omega t + ik_x x + ik_y y)\right],$$

where k_x and k_y are real horizontal wavenumbers, and show that

$$\omega^2 \int_a^b \rho|\boldsymbol{\xi}|^2\,dz = [\xi_z^*\,\delta\Pi]_a^b + \int_a^b \left(\frac{|\delta\Pi|^2}{\gamma p + \frac{B^2}{\mu_0}} - \frac{\left|\rho g \xi_z + \frac{B^2}{\mu_0} i k_y \xi_y\right|^2}{\gamma p + \frac{B^2}{\mu_0}} + \frac{B^2}{\mu_0} k_y^2 |\boldsymbol{\xi}|^2 - g\frac{d\rho}{dz}|\xi_z|^2 \right) dz,$$

where $z = a$ and $z = b$ are the lower and upper boundaries of the atmosphere, and $\delta\Pi$ is the Eulerian perturbation of total pressure. (Self-gravitation may be neglected.)

You may assume that the atmosphere is unstable if and only if the integral on the right-hand side can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, which are such that $[\xi_z^*\,\delta\Pi]_a^b = 0$. You may also assume that the horizontal wavenumbers are unconstrained. Explain why the integral can be minimized with respect to ξ_x by letting $\xi_x \to 0$ and $k_x \to \infty$ in such a way that $\delta\Pi = 0$.

Hence show that the atmosphere is unstable to disturbances with $k_y = 0$ if and only if

$$-\frac{d\ln\rho}{dz} < \frac{\rho g}{\gamma p + \frac{B^2}{\mu_0}}$$

at some point.

Assuming that this condition is not satisfied anywhere, show further that the atmosphere is unstable to disturbances with $k_y \neq 0$ if and only if

$$-\frac{d\ln\rho}{dz} < \frac{\rho g}{\gamma p}$$

at some point.

How does these stability criteria compare with the hydrodynamic stability criterion $N^2 < 0$?

4. *Waves in a rotating fluid*

Write down the equations of ideal gas dynamics in cylindrical polar coordinates (r, ϕ, z), assuming axisymmetry. Consider a steady, axisymmetric basic state in uniform rotation, with density $\rho(r, z)$, pressure $p(r, z)$ and velocity $\mathbf{u} = r\Omega\,\mathbf{e}_\phi$. Determine the linearized equations governing axisymmetric perturbations of the form

$$\mathrm{Re}\left[\delta\rho(r,z)\,e^{-i\omega t}\right] ,$$

etc. If the basic state is adiabatically stratified (i.e. $s =$ constant) and self-gravity may be neglected, show that the linearized equations reduce to

$$-i\omega\,\delta u_r - 2\Omega\,\delta u_\phi = -\frac{\partial W}{\partial r} ,$$

$$-i\omega\,\delta u_\phi + 2\Omega\,\delta u_r = 0 ,$$

$$-i\omega\,\delta u_z = -\frac{\partial W}{\partial z} ,$$

$$-i\omega W + \frac{v_\mathrm{s}^2}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\rho\,\delta u_r) + \frac{\partial}{\partial z}(\rho\,\delta u_z)\right] = 0 ,$$

where $W = \delta p/\rho$.

Eliminate $\delta\mathbf{u}$ to obtain a second-order partial differential equation for W. Is the equation of elliptic or hyperbolic type? What are the relevant solutions of this equation if the fluid has uniform density and fills a cylindrical container $\{r < a,\ 0 < z < H\}$ with rigid boundaries?

Answers to questions 1 and 2 may be submitted for marking.
Please send any comments and corrections to `gio10@cam.ac.uk`