Structure and Evolution of Stars

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[For Examples class 1.30-3.30pm Tuesday 26th November] Michaelmas Term, 2024

Examples Sheet III

1. In solar-like stars nuclear burning is dominated by the ppI and ppII chains

$$
\begin{array}{l}
{}^{1}\mathrm{H}({}^{1}\mathrm{H},\mathrm{e}^{+}\nu){}^{2}\mathrm{H}({}^{1}\mathrm{H},\gamma){}^{3}\mathrm{He}({}^{3}\mathrm{He},2{}^{1}\mathrm{H}){}^{4}\mathrm{He} \\
\qquad\qquad\qquad\qquad\qquad | \\
\qquad\qquad\qquad\qquad\qquad {}^{3}\mathrm{He}({}^{4}\mathrm{He},\gamma){}^{7}\mathrm{Be}(\mathrm{e}^{-},\nu){}^{7}\mathrm{Li}({}^{1}\mathrm{H},{}^{4}\mathrm{He}){}^{4}\mathrm{He}.
\end{array}
$$

The reaction rate between species i and j is

$$\frac{\lambda_{ij} n_i n_j}{1+\delta_{ij}}, \qquad (*)$$

where n_i is the number density of species i, δ_{ij} is the Kronecker delta and $\lambda_{ij} \propto \eta^2 \mathrm{e}^{-\eta}$ with $\eta = 42.48(AZ_i^2 Z_j^2 T_6^{-1})^{1/3}$, $A = A_i A_j/(A_i + A_j)$ is the reduced atomic mass of the two reacting nuclei, Z_i is the atomic number of species i and T_6 is related to temperature T by $T_6 = T/10^6\,\mathrm{K}$.

The beta decay of ^{7}Be is fast compared to all other reactions so that ^{7}Li is the predominant species of atomic mass 7 and all major species can be identified by $i \approx A_i$. Show that the temperature dependence of the rate r_{11} at the centre of the Sun, where $T_6 \approx 15$, of the reaction ${}^{1}\mathrm{H}({}^{1}\mathrm{H},\mathrm{e}^{+}\nu){}^{2}\mathrm{H}$ can be written as $r_{11} \propto T^{\alpha}$, where $\alpha = \frac{1}{3}(\eta - 2) \approx 4$. Also show that β and γ are approximately 16 (with $\gamma > \beta$) in the expressions $r_{33} \propto T^{\beta}$ and $r_{34} \propto T^{\gamma}$.

Show that the rate of change of protons obeys

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -\lambda_{11} n_1^2 - \lambda_{21} n_2 n_1 + \lambda_{33} n_3^2 - \lambda_{17} n_1 n_7,$$

and obtain the equivalent equations for n_2, n_3 and n_4.

At the centre of the Sun the characteristic timescale for ${}^{1}\mathrm{H}({}^{1}\mathrm{H},\mathrm{e}^{+}\nu){}^{2}\mathrm{H}$ is about $10^{10}\,\mathrm{yr}$ while that of ${}^{2}\mathrm{H}({}^{1}\mathrm{H},\gamma){}^{3}\mathrm{He}$ is about $1\,\mathrm{s}$. The characteristic timescale for n_3 to reach equilibrium is $\tau \approx 6 \times 10^5\,\mathrm{yr}$. By making an appropriate approximation, to be explained, show that

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} \approx -\frac{3}{2}\lambda_{11} n_1^2 + \lambda_{33} n_3^2 - \lambda_{17} n_1 n_7$$

and

$$\frac{\mathrm{d}n_3}{\mathrm{d}t} \approx \frac{1}{2}\lambda_{11} n_1^2 - \lambda_{33} n_3^2 - \lambda_{34} n_3 n_4$$

near the centre of the Sun.

Show further that $n_3 \approx n_{3\mathrm{e}}$ where

$$n_{3\mathrm{e}} = -\frac{\lambda_{34} n_4}{2\lambda_{33}} + \sqrt{\left(\frac{\lambda_{34} n_4}{2\lambda_{33}}\right)^2 + \frac{\lambda_{11} n_1^2}{2\lambda_{33}}}.$$

Consider a small perturbation of the form $n_3 = n_{3\mathrm{e}} + x$ about this equilibrium and linearise the evolution equation for n_3 to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\tau},$$

where $\tau = (2\lambda_{33} n_{3\mathrm{e}} + \lambda_{34} n_4)^{-1}$.

Estimate the temperature at which τ is comparable to the age of the Sun.

Sketch the abundances X_1 and X_3 of ^{1}H and ^{3}He as a function of radius in the Sun today.

2. A set of fully radiative stars has uniform mean molecular weight μ, constant opacity and energy generation rate $\epsilon = \epsilon_0 \rho T^{16}$, where ϵ_0 is constant. Radiation pressure is negligible. Use a simple homology argument to show that for this set of stars

$$L \text{ varies as } \mu^4 M^3,$$
$$T_\mathrm{c} \text{ varies as } \mu^{7/19} M^{4/19}$$
$$\text{and } R \text{ varies as } \mu^{12/19} M^{15/19},$$

where M is the star's mass, L its luminosity, R its radius and T_c its central temperature.

Hence show that the slope of the theoretical main sequence for such a set of stars is

$$-\frac{d \log L}{d \log T_\mathrm{eff}} = -\frac{76}{9},$$

where T_eff is the effective (surface) temperature.

Consider now two such sets of stars which differ in that one set is composed predominantly of hydrogen while the other is predominantly helium. By considering the ratio of luminosities at fixed effective temperature for the two sets of stars, or otherwise, show that the helium main sequence lies below and to the left of the hydrogen main sequence in a Hertzsprung–Russell diagram.

3. Indicate why in a fully convective star the equation of state may be taken to be $P = KT^{5/2}$ where K is a constant. Integrations for the atmospheric structure show that $K = Ag^\nu T_\mathrm{e}^{-\lambda}$ where A, ν and λ are constant, g is the surface gravity and T_e the effective temperature. Derive a luminosity–mass-radius relation in the form

$$\frac{L}{4\pi\sigma R^2} = C R^\alpha M^\beta \ ,$$

where C, α and β are constant and α and β depend solely on ν and λ. Show that, when $\nu = 3/4$, $T_{\rm e}$ is constant. In this case show that the time for a fully convective star to contract to radius R_s radiating its gravitational energy is

$$t = \frac{GM^2}{7\,(4\pi R_s^3 T_{\rm e}^4)\,\sigma}.$$

[*You may quote any properties of polytropes that you need.*]

4. A white dwarf may be approximated by a two-zone model. A helium interior is composed of a non-relativistic fully degenerate electron gas at constant temperature $T_{\rm c}$ with equation of state

$$P_{\rm e} = K\rho^{5/3}, \qquad (*)$$

where K is constant and $P_{\rm e}$ is the electron pressure. The very thin outer layers are composed of hydrogen gas in radiative equilibrium obeying the perfect-gas equation of state with negligible radiation pressure and with opacity given by Kramers' law in the form

$$\kappa = \kappa_0 \rho T^{-3.5}.$$

The transition between the inner and outer zones is defined to be where the total pressure given by the perfect-gas law is equal to the electron pressure given by equation $(*)$. Show that

(i) in the very thin outer layers of the white dwarf

$$P^2 = \frac{64}{51}\frac{\pi acGM}{\kappa_0 L}\left(\frac{R}{\mu}\right)T^{8.5} \equiv JT^{8.5},$$

where M and L are the mass and luminosity of the white dwarf,

(ii) the temperature at the transition $T_{\rm tr}$ is

$$T_{\rm tr} = \left(\frac{R}{\mu}\right)^{10/7} K^{-6/7} J^{-2/7}$$

and (iii) the luminosity of the white dwarf is

$$L = \frac{64}{51}\frac{\pi acGM}{\kappa_0}\left(\frac{\mu}{R}\right)^4 K^3\, T_{\rm c}^{3.5}.$$

By taking plausible numerical values, which should be stated, estimate to order of magnitude the temperature in the interior of the white dwarf.

Comment on the source of energy for the white dwarf's luminosity and estimate, in years, to order of magnitude, the cooling time scale of the white dwarf.

[$K \approx 10^{13}\,\mathrm{dyne\,cm^3 g^{-5/3}}$, $\kappa_0 \approx 4\times 10^{24}\,\mathrm{cm^5 g^{-2} K^{7/2}}$.]

5. A model of a red giant consists of an isothermal degenerate helium core of mass M_1 and radius R_1. These are related by the mass-radius relation $M_1^{1/3}R_1 = A =$ constant. At the core boundary there is a thin hydrogen-burning shell which generates the entire luminosity L. Above the shell there is a radiative envelope which contains a negligible amount of mass and above that is a convective envelope with a significant amount of mass. If the opacity is given by the power law $\kappa = \kappa_0\rho^n/T^m$, with n and m constant, show that, if radiation pressure is neglected, the relation between P and T in the radiative envelope is

$$P = C\left(T^{4+m+n} + T_0^{4+m+n}\right)^{1/(n+1)},$$

where

$$C = \left[\frac{16\pi acGM_1}{3\kappa_0 L}\left(\frac{R}{\mu}\right)^n \frac{(n+1)}{(n+m+4)}\right]^{1/(n+1)}$$

and T_0 is an appropriate constant of integration. Show that if T_b is the temperature at the base of the convective envelope, then

$$T_0 = T_\mathrm{b}\left\{\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{4+m+n}{n+1}\right) - 1\right\}^{+1/(4+n+m)}.$$

Show that in regions near the shell, well below the inner boundary of the convective envelope where $T \gg T_\mathrm{b}$ and hence $T \gg T_0$, the dependence of temperature on radius r is approximately given by

$$T = \frac{\mu}{R}\frac{GM_1(n+1)}{(4+n+m)r}.$$

Use this to show that, for the case in which $n = 1$, $m = 3$ and the energy generation rate is given by $\epsilon = \epsilon_0\rho T^{10}$ with ϵ_0 constant,

$$L = \frac{4\pi}{13}C^2\epsilon_0\left(\frac{\mu}{R}\right)^2\left(\frac{\mu GM_1}{4R}\right)^{16}\frac{1}{R_1^{13}}.$$

Hence derive the relation between core mass and luminosity in the form

$$L \propto M_1^{32/3}.$$

What happens if mass is removed from the stellar envelope?