Structure and Evolution of Stars

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[For Examples class 1.30-3.30pm Tuesday 28th January 2025] Michaelmas Term, 2024

Examples Sheet IV

1. Two stars of equal surface temperatures form an eclipsing binary system. The orbit is circular. If the stars are spherical and limb-darkening can be ignored, show that both eclipses are the same depth (in magnitudes) and that this depth cannot exceed a certain amount.

2. In a single-lined spectroscopic binary with a circular orbit the only measurable parameters are the orbital period P and the semi-amplitude K_1 of the radial velocity of star 1. If the (unknown) inclination of the orbit is i, show that the mass function defined as

$$F\left(M_1, M_2, i\right) = \frac{M_2^3 \sin^3 i}{\left(M_1 + M_2\right)^2}$$

can be determined from the observations. Here M_1 and M_2 are the masses of star 1 and star 2 respectively. Show that for given i, the observations give a lower limit to M_2. How might we set limits on i?

3. Two stars of masses M_1 and M_2 move in circular orbits about their centre of gravity. What is (i) the orbital angular momentum h_1 of star 1 and (ii) the total orbital angular momentum h, in terms of the masses and the orbital period P?

Wind from star 1 reduces M_1 at a rate which is steady and slow compared with P. The wind leaves the stellar surface in a spherically symmetric manner and can be assumed not to interact with star 2. Justify briefly the equation

$$\dot{h} = \frac{h_1}{M_1} \dot{M}_1$$

for the evolution of total orbital angular momentum and show that this implies

$$P \propto \left(M_1 + M_2\right)^{-2}.$$

Now suppose that (a) a constant fraction f of the mass lost from star 1 is accreted by star 2, (b) the remaining $1 - f$ escapes to infinity with the same specific angular momentum as before and (c) the intrinsic spin of both components remains negligible compared to the orbital angular momentum. Show that the variation of P is now given by

$$P \propto M_1^{-3f} M_2^{-3} \left(M_1 + M_2\right)^{-2}.$$

Discuss qualitatively the validity of each of assumptions (a) to (c) and in particular their possible dependence on orbital separation (relative to star size) and on wind velocity (relative to orbital velocity).

Finally, suppose that the period is so long, perhaps $10^6 - 10^7$ yr, that the mass loss by stellar wind takes place in a time which is *short* compared with the orbital period. Without detailed calculation, what would you expect the effect on the orbit to be?

4. In a semi-detached binary, with conservative Roche lobe overflow (RLOF), show that the radius a of the orbit satisfies

$$a \propto \frac{(1+q)^4}{q^2},$$

where q is the mass ratio of the two components (loser/gainer). In a certain range of mass ratios, the radius of the Roche lobe around the loser can be approximated by

$$R_{\mathrm{L}} \approx 0.4 a q^{2/9}.$$

Show that as the loser, of mass m, transfers mass to its companion, its Roche-lobe radius changes at a rate

$$\frac{d \log_e R_{\mathrm{L}}}{dt} = \alpha \frac{d \log_e m}{dt},$$

where

$$\alpha = \tfrac{20}{9}\left(q - \tfrac{4}{5}\right).$$

A star of mass M and age t has a radius R which changes in response to internal nuclear evolution and to variation in mass (provided the variation is slow), according to

$$\log_e R = \beta \log_e M + t/t_{\mathrm{nuc}},$$

where β, the slope of the ZAMS radius-mass relation and t_{nuc}, the nuclear timescale, can be taken as constant. As long as $R < R_{\mathrm{L}}$ the mass m remains constant but once $R > R_{\mathrm{L}}$ mass starts to flow at a rate given by

$$\frac{d \log_e M}{dt} = -\frac{1}{t_{\mathrm{dyn}}} \log_e \frac{R}{R_{\mathrm{L}}},$$

where t_{dyn} is the dynamical timescale, also constant $(t_{\mathrm{dyn}} \ll t_{\mathrm{nuc}})$.

Define $f = \log_e (R/R_{\mathrm{L}})$. As long as f is negative show that it satisfies the differential equation

$$\frac{df}{dt} = \frac{1}{t_{\mathrm{nuc}}}$$

and find a corresponding first-order linear differential equation satisfied by f when it is positive. Show that, as long as $\beta > \alpha$, f tends to a small constant positive value,

$$f \to \frac{1}{\beta - \alpha} \frac{t_{\mathrm{dyn}}}{t_{\mathrm{nuc}}},$$

with steady mass transfer on a nuclear timescale, but that if $\beta < \alpha$ then f grows exponentially on a dynamical timescale.

On the lower main sequence $\beta \approx 1$, while on the upper main sequence $\beta \approx 0.5$. Find the corresponding ranges of initial mass ratio q_0 for which mass transfer can proceed steadily, on a nuclear timescale, once the primary has filled its Roche lobe.

5. For mass ratio q between 0.2 and 2 the ratio of Roche-lobe radius to orbital separation is given sufficiently accurately by

$$\frac{R_{\rm L}}{a} = 0.38q^{0.25}.$$

When a star transfers mass conservatively to a companion in a close binary show that the radius of the loser is a minimum when $q = \frac{7}{9}$.

Such a loser is a white dwarf whose radius R_* is given in terms of its mass M by

$$\frac{R_*}{R_\odot} = 0.01\left(\frac{M}{M_\odot}\right)^{-\frac{1}{3}}, \qquad (\dagger).$$

Show that Roche-lobe overflow must be on a rapid hydrodynamic timescale if $q > \frac{17}{27}$ but that it can be on a slower timescale otherwise.

A binary system with a period of 0.1 d consists of two white dwarfs of different mass, both in the range $0.2 - 1\,M_\odot$, for which you may assume (†) applies. Given that the Sun would fill its Roche lobe in a binary with period of about 0.3 d, estimate the period at which the double-white-dwarf binary would fill its Roche lobe. Which component would reach its lobe first? What physical process or processes might bring about the necessary decrease in period within the Galactic lifetime? What would you expect to be the outcome of Roche-lobe overflow if the white dwarfs' masses are (a) $0.2 + 0.5\,M_\odot$, (b) $0.5 + 1.0\,M_\odot$ and (c) $0.6 + 0.7\,M_\odot$.

6. A supergiant of C/O core mass $M_{\rm c}$ and envelope mass $M_{\rm env}$, for which the binding energy may be expressed as

$$E_{\rm bind} \approx \frac{2GM_{\rm env}^2}{R_{\rm G}},$$

where $R_{\rm G}$ is a fiducial radius defined by

$$\frac{R_{\rm G}}{R_\odot} \approx 1{,}000\left(\frac{M_{\rm c}}{M_\odot}\right)^2\left(\frac{M_{\rm env}}{M_\odot}\right)^{-\frac{1}{3}},$$

is in a binary with a C/O white dwarf of mass $M_{\rm wd}$.

The giant fills its Roche lobe and dynamical mass transfer leading to common-envelope evolution ensues. Show that, if the common envelope efficiency is α_{ce}, the final separation of the cores when the envelope has been lost is a_{f}, where

$$\frac{a_{\mathrm{f}}}{R_{\mathrm{G}}} \approx \frac{\alpha_{\mathrm{ce}}}{2} \frac{M_{\mathrm{c}} M_{\mathrm{wd}}}{M_{\mathrm{env}}^2}$$

in the limit $a_{\mathrm{f}} \ll a_{\mathrm{i}}$, the initial separation.

The radius of a white dwarf of mass M_{wd} can be approximated by

$$\frac{R_{\mathrm{wd}}}{R_{\odot}} \approx 0.01 \left(\frac{M_{\odot}}{M_{\mathrm{wd}}} \right)^{\frac{1}{3}}$$

and the radius of the hot giant core by

$$R_{\mathrm{c}} \left(M_{\mathrm{c}} \right) \approx 5 R_{\mathrm{wd}} \left(M_{\mathrm{c}} \right).$$

The spiralling cores coalesce if $a_{\mathrm{f}} \leq 3 \max \left(R_{\mathrm{wd}}, R_{\mathrm{c}} \right)$. Estimate the minimum envelope mass M_{crit} required for the cores to coalesce if $M_{\mathrm{c}} = 0.6\, M_{\odot}$, $M_{\mathrm{wd}} = 0.9\, M_{\odot}$ and $\alpha_{\mathrm{ce}} = 1.0$.

Suppose the process of coalescence heats the degenerate white dwarf and the supergiant core to a temperature at which carbon burning $^{12}\mathrm{C} + ^{12}\mathrm{C} \rightarrow ^{24}\mathrm{Mg}$, (13.93 MeV per reaction) can ignite. Estimate the total nuclear energy that can be released and compare it with the binding energy of the white dwarf which may be modelled as an $n = 3/2$ polytrope.

Comment on the result for M_{env} in the range from well below to well above M_{crit}.