Quantum Fluids: Example Sheet 3

Problem 1. A vortex in a trapped cloud. Assume that the extent of the axisymmetrical trapped condensate in $z-$ direction is much greater than the healing length, ξ_0, in the centre of the cloud ($R_z \ll \xi_0$) and the density on z-axis can be approximated by $n_0(z) = n(0)(1 - z^2/R_z^2)$.
(a) Show that the energy of the vortex is approximated by

$$\mathcal{E} = \frac{4\pi\hbar^2 n(0)}{3m} R_z \ln\left[0.671\frac{R}{\xi_0}\right] dz$$

where the kinetic energy change due to vertical gradients is neglected and R is the size of the cloud in $xy-$plane.
(b) Calculate the total angular momentum of the three-dimensional cloud of the condensate in a parabolic trap with a vortex along the $z-$ axis, assuming that the healing length is much less than R_z.

Problem 2. Nonlocal potential.

The GP equation can be modified in the spirit of a density - functional approach to describe some aspects of superfluid helium. The idea is that the interaction energy term $\sim |\psi|^4$ in the GP energy functional is replaced by the interaction energy

$$W_c(\rho) = \frac{1}{M^2}\int \left[\frac{1}{2}\int \rho(\mathbf{x})\tilde{V}(|\mathbf{x}-\mathbf{x}'|)\rho(\mathbf{x}')\, d\mathbf{x}' + \frac{W_1}{2+\gamma}\rho^{2+\gamma}\right] d\mathbf{x},$$

where γ and W_1 are given and $\tilde{V}(|\mathbf{x}-\mathbf{x}'|)$ is the two-body interaction potential that can be taken as

$$\tilde{V}(|\mathbf{x}-\mathbf{x}'|) = \tilde{V}(r) = (\alpha + \beta A^2 r^2 + \delta A^4 r^4)\exp(-A^2 r^2),$$

where A, α, β and δ are parameters that can be chosen to give excellent agreement with the experimentally determined dispersion curve.
(a) Derive the GP equation with the nonlocal potential given above.
(b) Introduce the chemical potential μ and cast the resulting nonlocal GP equation into the dimensionless form

$$-2i\frac{\partial \Psi}{\partial t} = \nabla^2\Psi + \Psi\left[1 - \int |\Psi(\mathbf{x}')|^2 V(|\mathbf{x}-\mathbf{x}'|)\, d\mathbf{x}' - \chi|\Psi|^{2(1+\gamma)}\right]. \quad (1)$$

What are the nondimensional units? Write down the expression for χ. How is V related to $\tilde{V}$?
(c) What is the normalisation condition on V?
(d*) Find the dispersion relation for (1).

(e*) Show that the equation for the amplitude of the steady straight line vortex $R(r)$ is given by

$$R''(r) + \frac{1}{r}R'(r) - \frac{R(r)}{r^2} + R(r) - \chi R(r)^{2(1+\gamma)+1}$$
$$= \frac{2\pi^{3/2}}{A} R(r) \int_0^\infty R(r')^2 \exp(-A^2(r^2 + r'^2)) \left[g_0 I_0(\sigma) - g_1 I_1(\sigma)\right] r'\, dr',$$

where I_n is the modified Bessel function of order n. What are g_0, g_1 and σ?

Problem 3. Consider the following dimensionless GLE and the rate equation:

$$i\frac{\partial \psi}{\partial t} = \left(-\nabla^2 + |\psi|^2 + gn + i(n-1)\right)\psi \tag{2}$$

$$\gamma\frac{\partial n}{\partial t} = -\left(1 + b|\psi|^2 + cn\right)n + p \tag{3}$$

substitute the equilibrium values of the reservoir into GPE and let $\psi = \Psi\exp(-i\mu_s t)$, where μ_s is the chemical potential, then using the Madelung transformation ($\psi = \sqrt{\rho}\exp(iS)$) find an analytical approximation of velocity $\mathbf{u} = \nabla S$ in terms of μ_s in the limit of $r \to \infty$. Assume that you have a radially symmetric Gaussian pumping profile.

Problem 4. We can model the left- and right-circular polariton states ψ_L and ψ_R in exciton-polariton condensates using weakly-interacting dilute Bose gas model with energy given by

$$\begin{aligned} E &= \frac{\hbar^2|\nabla\psi_L|^2}{2m} + \frac{\hbar^2|\nabla\psi_R|^2}{2m} + \frac{U_0}{2}\left(|\psi_L|^2 + |\psi_R|^2\right)^2 \\ &- 2U_1|\psi_L|^2|\psi_R|^2 + \Omega_B\left(|\psi_L|^2 - |\psi_R|^2\right) \\ &+ J_1\left(\psi_L^\dagger\psi_R + H.c\right) + J_2\left(\psi_L^\dagger\psi_R + H.c.\right)^2 \end{aligned}$$

where we introduced the tendency to biexciton formation via U_1 and the action of the applied magnetic field Ω_B. J_1 and J_2 represent the breaking of symmetry.

(a) Write down the corresponding equations on the time evolution of ψ_L and ψ_R using the Hamiltonian formulation $i\hbar\psi_i = \partial E/\partial\psi_i^*$, where $i = L, R$.

(b) Let $J_2 = 0$ and modify the equations in (a) by adding linear effective gain and nonlinear dissipation appropriate for exciton-polariton condensates. Show that in the dimensionless form the resulting equations can be written as

$$2i\partial_t\psi_L = \left(-\nabla^2 + |\psi_L|^2 + (1-u_a)|\psi_R|^2 + \frac{\Delta}{2} + i(\alpha - \sigma|\psi_L|^2)\right)\psi_L + J\psi_R,$$

and similarly for ψ_R. Write down the expressions for dimensionless parameters.

(c) Neglect the spatial variations and write down the ordinary differential equations on θ, R and z where $\psi_{L,R}(t) = \sqrt{\rho_{L,R}(t)}\exp[i(\phi(t) \pm \theta(t)/2)]$, $R(t) = (\rho_L(t) + \rho_R(t))/2$, $z(t) = (\rho_L(t) - \rho_R(t))/2$.

(d) Show that in the so-called "Josephson regime" where $u_a R \gg J$ and $z \ll R$ the steady state of R is $R = \alpha/\sigma$ and the ODEs found in (c) reduce to the second order single ODE on a driven damped pendulum. Discuss the behaviour of the system depending on the system parameters.