3C7b **Perturbation Methods: Examples Sheet 2** Michaelmas 2020

A star ∗ denotes a question, or part of a question, that should not be done at the expense of unstarred questions (e.g. you might like to miss them out first time through the sheet). You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to S.J.Cowley@maths.cam.ac.uk.

If you would like questions 3, 6, 8 and 9(b) marked in advance of the second Examples Class on 30 November, please note the following:

- *the deadline for handing in your work is midnight on Thursday 26 November;*

- *please submit your work electronically* **in PDF format** *via the Perturbation Methods Moodle site (scroll down to the submission link for Examples Sheet 2 in the Examples Classes section);*

- *please name the PDF file* CRSid-PM-Sheet-2.pdf, *and write your full name and CRSid on your work.*

∗1. *1998, Paper 46, Q2.* The integral $\mathcal{E}_n(z)$ is defined by

$$\mathcal{E}_n(z) = \int_z^\infty \frac{\mathrm{e}^{-t^2}}{t^{2n}} \, \mathrm{d}t \,,$$

where z is a complex number $(-\pi < \arg z \leqslant \pi)$, n is an integer, and the integration contour starts from $t = z$ and extends to $t = \infty$ in the sector $|\arg(t)| < \pi/4$.

(a) For $z = n^{\frac{1}{2}}\mathrm{e}^{i\theta}$, and for all values of θ in the range $-\pi < \theta \leqslant \pi$, obtain the appropriate leading-order asymptotic expansion of $\mathcal{E}_n(z)$ as $n \to \infty$. *Hints:* (i) see the figure below, and (ii) while you should find that the *form* of leading-order asymptotic expansion changes at certain values of θ, you should *not* examine in detail the nature of the change at these 'transition' values of θ.
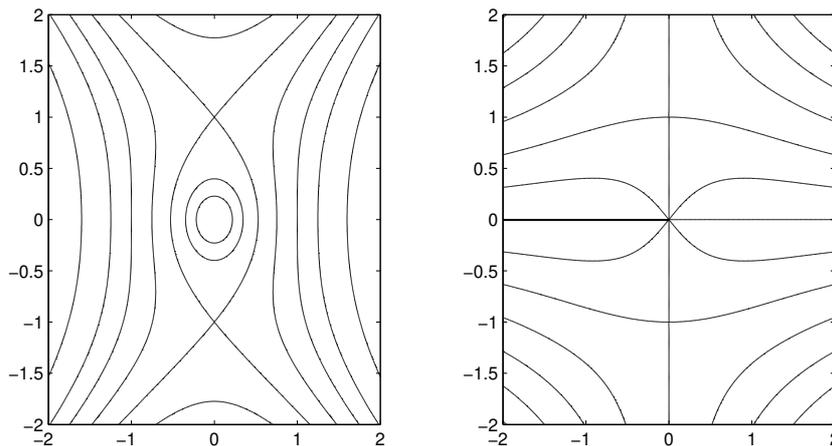
Contours of constant $Re(w^2 + 2\log(w))$ and $Im(w^2 + 2\log(w))$ for complex w.
The branch cut is taken down the negative real axis.

(b) It is possible to show by repeated integration by parts (do *not* prove this) that

$$\tfrac{1}{2}\pi^{1/2}(1 - \mathrm{erf}(z)) \equiv \mathcal{E}_0(z) = \frac{\mathrm{e}^{-z^2}}{2z}\left(\sum_{r=0}^{n-1} \frac{(-)^r(2r)!}{(2z)^{2r}r!}\right) + \mathcal{R}_n(z)\,,$$

where the remainder $\mathcal{R}_n(z)$ is given by

$$\mathcal{R}_n(z) = \frac{(-)^n(2n)!}{2^{2n}n!}\,\mathcal{E}_n(z)\,.$$

An 'optimal' asymptotic series for $\mathcal{E}_0(z)$ as $|z| \to \infty$ can be formed by truncating the series at the smallest term. For a given value of z, for what approximate value of n should the series be truncated? For this 'optimal' truncation, estimate the size of the remainder. Briefly discuss the dependence of the remainder on $\arg z$ $(-\pi < \arg z \leqslant \pi)$.

Stirling's asymptotic approximation for the Gamma function may prove useful:

$$(n-1)! = \Gamma(n) \sim \left(\frac{2\pi}{n}\right)^{\frac{1}{2}} n^n \, \mathrm{e}^{-n}\,.$$

2. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \,,$$

and is subject to boundary conditions $y = 0$ at $x = 0$ and $y = e^{-1}$ at $x = 1$. Find two terms in the outer approximation, applying only the boundary condition at $x = 1$. Next find two terms in the inner approximation for the $\mathrm{ord}(\epsilon)$ boundary layer near to $x = 0$; apply only the boundary condition at $x = 0$. Finally determine the constants of integration in the inner approximation by matching.

3. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + x^{1/2}y' + y = 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \,,$$

and is subject to the boundary conditions $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$. First find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.

4. The function $y(x;\epsilon)$ satisfies

$$(x + \epsilon y)y' + y = 1 \quad \text{with} \quad y(1) = 2 \,.$$

Find $y(0)$ correct to $\mathrm{ord}(1)$.

5. The function $y(x,\epsilon)$ satisfies

$$\epsilon y'' + yy' - y = 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \,,$$

and is subject to the boundary condition $y = 0$ at $x = 0$ and $y = 3$ at $x = 1$. Assuming that there is a boundary layer only near $x = 0$, find the leading order terms in the outer and inner approximations and match them.

Suppose now the boundary conditions are replaced by $y = -\frac{3}{4}$ at $x = 0$ and $y = \frac{5}{4}$ at $x = 1$. Show that the boundary layer moves to an intermediate position which is determined by the property of the inner solution that y jumps within the boundary layer from $-M$ to M, for some value M. Find the leading order matched asymptotic expansions.

6. Consider the following problem which has an outer, an inner and an inner-inner inside the inner

$$x^3 y' = \epsilon\left((1+\epsilon)x + 2\epsilon^2\right)y^2 \quad \text{in} \quad 0 < x < 1 \,,$$

with $y(1) = 1 - \epsilon$. Calculate two terms of the outer, then two of the inner, and finally one for the inner-inner. At each stage find the rescaling required for the next layer by examining the non-uniformity of the asymptoticness in the current layer.

7. The function $y(x;\epsilon)$ satisfies
$$(\epsilon + x)y' = \epsilon y \quad \text{with} \quad y(1) = 1 \,.$$
Find $y(0)$ correct to $\mathrm{ord}(\epsilon^2)$.

8. The function $f(r,\epsilon)$ satisfies the equation

$$f_{rr} + \frac{2}{r}f_r + \frac{1}{2}\epsilon^2(1 - f^2) = 0 \quad \text{in} \quad r > 1 \,,$$

and is subject to the boundary conditions

$$f = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad f \to 1 \quad \text{as} \quad r \to \infty \,.$$

Using the asymptotic sequence 1, ϵ, $\epsilon^2 \ln\frac{1}{\epsilon}$, ϵ^2, obtain asymptotic expansions for f at fixed r as $\epsilon \searrow 0$ and at fixed $\rho = \epsilon r$ as $\epsilon \searrow 0$. Match the expansions using the intermediate variable $\eta = \epsilon^\alpha r$ with $0 < \alpha < 1$.

Hint. You may quote that the solution to the equation

$$y_{xx} + \tfrac{2}{x}y_x - y \;\; = \;\; \frac{e^{-2x}}{x^2} \,,$$

subject to the condition $y \to 0$ as $x \to \infty$, is

$$y \;\; = \;\; A\frac{e^{-x}}{x} \;\; + \;\; \frac{1}{2x}\int_x^\infty \frac{e^{-x-t} - e^{x-3t}}{t}\, dt \,,$$

with A a constant. Further as $x \to 0$

$$y \;\; \sim \;\; \frac{2A + \ln 3}{2x} \;\; + \;\; \ln x - A + \gamma + \tfrac{1}{2}\ln 3 - 1 \,.$$

9. *2014, Paper 74, Q3.*

   (a) The function $y(x)$ satisfies the differential equation

   $$(1+\varepsilon)x^2 y' = (1-\varepsilon)\varepsilon\, xy^2 - (1+\varepsilon)\varepsilon\, x + \varepsilon\, y^3 + 2\varepsilon^2\, y^2 \quad \text{in} \quad 0 \leqslant x \leqslant 1\,,$$

   where $0 < \varepsilon \ll 1$. If $y(1) = 1$, calculate three terms of the outer solution of y. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for the inner solution.
   [*Hint: The general solution to*

   $$\xi^2 g' - \left(\frac{3\xi}{2+\xi}\right) g = -\left(\frac{\xi}{2+\xi}\right)^{\frac{3}{2}}$$

   is

   $$g(\xi) = \frac{(1 + k\xi)\xi^{\frac{1}{2}}}{(2+\xi)^{\frac{3}{2}}}\,,$$

   for some constant k.]

   (b) Using matched asymptotic expansions find the value of $z'(0)$ to leading order if $z(1) = e^{-1}$ and $z(x)$ satisfies the equation
   $$(x - \varepsilon z)z' + xz = e^{-x}\,.$$

∗10. The function $\varphi(y,t)$ satisfies the partial differential equation

$$\frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\varphi}{3y^2}\right) = \frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\,,$$

where it is supposed that

$$
\begin{aligned}
\varphi(y,0) &= \varphi_0 && \text{for} \quad y > 0\,,\\
\varphi(y,t) &\to \varphi_0 && \text{as} \quad y \to \infty\,,\\
\varphi(y_0(t),t) &= 0 && \text{for} \quad t > 0\,,
\end{aligned}
$$

and $y_0(t)$ is to be determined. For $0 < t \ll 1$,

   (i) seek two terms of a series expansion in powers of t for $y = O(1)$;
   (ii) seek the leading-order term of a series expansion for $y = O(t^\mu)$, where μ is to be determined.

Is the solution that you have found in the $y = O(t^\mu)$ region uniformly valid? Identify a further rescaling of the form $y - \lambda t^\mu = O(t^\nu)$ where λ and ν are to be identified. Derive the leading-order governing equation in this region.