3P1b **Quantum Field Theory: Example Sheet 2** Michaelmas 2024

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1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \ . \tag{1}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x\ T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \ , \tag{2}$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x) \ .$$

2. For a real scalar field with Lagrangian (1), show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x) \, .$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

3. Let $\phi(x)$ be a real scalar field in the Heisenberg picture with Lagrangian (1). Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle$ satisfy

$$\langle 0| \phi(x) |p\rangle = e^{-ip\cdot x} \ .$$

4*. In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2}\epsilon_{ijk} \int d^3x\ \left(x^j T^{0k} - x^k T^{0j}\right) \ .$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = \frac{i}{2}\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \left(p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j}\right) a_{\vec{p}} \ .$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum in its rest frame).

5. Show that the time ordered product $\mathrm{T}(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

6*. The Schwinger-Dyson equation states that

$$\begin{aligned}\left(\Box_x + m^2\right)\langle\phi_x\phi_1\dots\phi_n\rangle = &\langle\mathcal{L}'_{\text{int}}[\phi_x]\phi_1\dots\phi_n\rangle \\ &- i\sum_{j=1}^{n}\delta^{(4)}(x-x_j)\langle\phi_1\dots\phi_{j-1}\phi_{j+1}\dots\phi_n\rangle\ ,\end{aligned} \tag{3}$$

where $\phi_j \equiv \phi(x_j)$ and $\phi_x \equiv \phi(x)$. Recall that the brackets stand for a shorthand of time-ordering, i.e.,

$$\langle\phi_1\dots\phi_n\rangle \equiv \langle\Omega|\,\mathrm{T}\left(\phi_1\dots\phi_n\right)|\Omega\rangle\ ,$$

and for simplicity we are assuming that the interacting part of the Lagrangian ($\mathcal{L}_{\text{int}}$) does not include derivatives of the fields.

In lectures we showed (3) for two fields for a QFT that is local and causal. Derive explicitly (3) for three fields by using the same assumptions.

7. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi\ .$$

Calculate the amplitude for meson decay $\phi \to \psi\bar{\psi}$ to leading order in g using the LSZ formula and the Schwinger-Dyson equation.

Show that the amplitude is only non-zero for $m > 2M$ and explain the physical interpretation of this condition using conservation laws.

8. *Optional: The interaction picture (Srednicki 9.5). See also Sec 3.1 and 3.7 of DT.*

In this problem, we will derive a formula for $\langle\Omega|\,\mathrm{T}\left(\phi_1\dots\phi_n\right)|\Omega\rangle$ using time-dependent perturbation theory for an interacting theory involving a single massive scalar field.

Suppose we have a Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\ , \tag{4}$$

and $\mathcal{H}_1$ is a function of $\pi(0,\vec{x})$ and $\phi(0,\vec{x})$ and their spatial derivatives.

The ground state of the whole system is $|\Omega\rangle$, and a constant is added to the Hamiltonian such that $H|\Omega\rangle = 0$. The ground state of H_0 is given by $|0\rangle$ and a constant is also added such that $H_0|0\rangle = 0$. The Heisenberg picture field is

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}\ . \tag{5}$$

We also define the interaction-picture field as

$$\phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}\ . \tag{6}$$

(a) Show that $\phi_I(x)$ obeys the Klein-Gordon equation, and hence is a free field.

(b) Show that

$$\phi(x) = U^\dagger(t)\phi_I(x)U(t) \ ,$$

where $U(t) = e^{iH_0 t}e^{-iHt}$ is unitary.

(c) Show that $U(t)$ obeys the differential equation

$$i\frac{d}{dt}U(t) = H_I(t)U(t) \ ,$$

where $H_I(t) = e^{iH_0 t}H_1 e^{-iH_0 t}$ is the interaction Hamiltonian in the interaction picture.

(d) $\mathcal{H}_1$ was specified by a particular function of the fields $\pi(0,\vec{x})$ and $\phi(0,\vec{x})$. Show that $\mathcal{H}_I(t)$ is given by the same function of the interaction-picture fields $\pi_I(t,\vec{x})$ and $\phi_I(t,\vec{x})$.

(e) Show that, for $t > 0$,

$$U(t) = \mathrm{T}\exp\left(-i\int_0^t dt' H_I(t')\right)$$

is a solution to the differential equation (7). What is the comparable expression for $t < 0$?

(f) Define $U(t_2,t_1) := U(t_2)U^\dagger(t_1)$. Show that, for $t_2 > t_1$,

$$U(t_2,t_1) = \mathrm{T}\exp\left(-i\int_{t_1}^{t_2} dt' H_I(t')\right)$$

What is the comparable expression for $t_1 > t_2$?

(g) For any time ordering, show that

$$\begin{aligned} U(t_3,t_1) &= U(t_3,t_2)U(t_2,t_1) \ , \\ U^\dagger(t_1,t_2) &= U(t_2,t_1) \ , \end{aligned}$$

and

$$\begin{aligned} U^\dagger(t,0) &= U^\dagger(\infty,0)U(\infty,t) \ , \\ U(t,0) &= U(t,-\infty)U(-\infty,0) \ . \end{aligned}$$

(h) Replace H_0 with $(1-i\epsilon)H_0$, and show that $\langle\Omega| U^\dagger(\infty,0) = \langle\Omega|0\rangle\langle 0|$ and that $U(-\infty,0)|\Omega\rangle = |0\rangle\langle 0|\Omega\rangle$.

(i) Show that

$$\phi(x_n)\ldots\phi(x_1) = U^\dagger(t_n,0)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})\ldots U(t_2,t_1)\phi_I(x_1)U(t_1,0) \ ,$$

and

$$\begin{aligned} \langle\Omega|\phi(x_n)\ldots\phi(x_1)|\Omega\rangle = |\langle\Omega|0\rangle|^2 \langle 0| U(\infty,t_n)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})\ldots \\ U(t_2,t_1)\phi_I(x_1)U(t_1,-\infty)|0\rangle \ . \end{aligned}$$

(j) Show that

$$\langle\Omega|\,\mathrm{T}(\phi(x_n)\ldots\phi(x_1))\,|\Omega\rangle = |\,\langle\Omega|0\rangle\,|^2\,\langle 0|\,\mathrm{T}(\phi_I(x_n)\ldots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I})\,|0\rangle\ \ .$$

(k) Show that

$$|\,\langle\Omega|0\rangle\,|^2 = \left(\langle 0|\,\mathrm{T}e^{-i\int d^4x\mathcal{H}_I}\,|0\rangle\right)^{-1}\ \ .$$

Thus we have

$$\langle\Omega|\,\mathrm{T}(\phi(x_n)\ldots\phi(x_1))\,|\Omega\rangle = \frac{\langle 0|\,\mathrm{T}(\phi_I(x_n)\ldots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I})\,|0\rangle}{\langle 0|\,\mathrm{T}e^{-i\int d^4x\mathcal{H}_I}\,|0\rangle}\ \ .$$

The right-hand side of this equation involves only interaction-picture fields, hence one can Taylor expand the exponentials and use free-field theory to compute the resulting correlation functions.