QFT: Decay Rates and Cross Sections

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So far, we have considered transition amplitudes between $|i\rangle$ and $|f\rangle$ asymptotic states of definite momententum: the probability is expressed in terms of the transition amplitude part of the S-matrix

$$\langle f|(S-1)|i\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_i - \sum_{r=1}^{n} q_r), \tag{1}$$

for n final state particles. p_i is the total 4-momentum of the initial state. The probability of transition for $i \to f$ will be

$$P = \frac{|\langle f|S-1|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}. \tag{2}$$

Examining Eq. 1, we see that there will be *two* momentum preserving delta functions in P, which is one too many. Really, this has come about because we have pretended that the external states are *pure momentum eigenstates.* This is an approximation: they are really *a very sharply peaked superposition of momentum eigenstates.* When we take this fact into account, it ends up absorbing the extraneous delta function.

Cross Sections

Now, we consider 2 particle beams colliding. The kinematics is depicted in Fig. 1. The initial state is depicted in the rest frame of particle 1 in Fig. 2, with the incoming beam of particle 2. See the Standard Model course next term (or

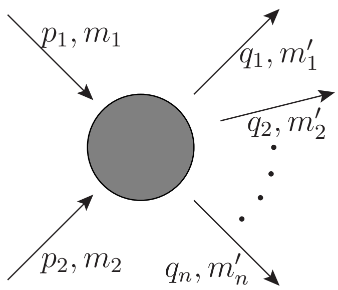

Figure 1: Kinematics of $2 \to n$ scattering.

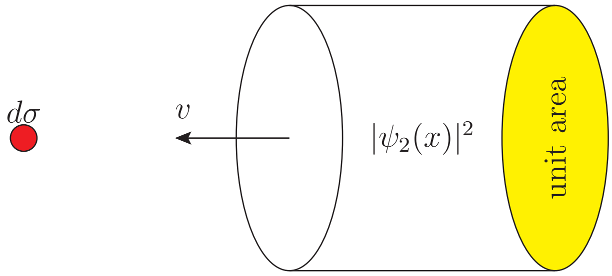

Figure 2: In the rest frame of initial particle 1, whose strength of interaction is as if it presents an effective cross-sectional area ($d\sigma$) for scattering into f.

Peskin and Schroeder) for a derivation of the result

$$d\sigma = \frac{(2\pi)^4}{\mathcal{F}} \delta^4(p_1 + p_2 - \sum_{i=1}^{n} q_i)|\mathcal{M}|^2, \quad (3)$$

where $\mathcal{F} = 4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}$ is known as the flux factor[1].

In order to find the integrated cross-section for $i \to f$, we must sum over the possible momenta of final states in the usual Lorentz invariant manner

$$\boxed{\sigma = \frac{1}{\mathcal{F}} \int dp_f \; |\mathcal{M}|^2} \quad (4)$$

where we have defined the total 4-momentum conserving integral over the final state momenta p_f

$$\int dp_f \equiv (2\pi)^4 \int \left(\prod_{r=1}^{n} \frac{d^3 q_r}{(2\pi)^3 2E_{q_r}} \right) \delta^4 \left(p_i - \sum_{r=1}^{n} q_r \right). \quad (5)$$

However, sometimes we may wish to obtain the *differential cross-section*, in order to examine its behaviour as a function of a kinematical variable.

2 to 2 Scattering

2 to 2 scattering is an important example: let us examine the cross section with respect to variations of Mandlestam variable

$$t \equiv (p_1 - q_1)^2 = m_1^2 + {m'_1}^2 - 2E_{p_1}E_{q_1} + 2\mathbf{p_1}.\mathbf{q_1} \Rightarrow \frac{dt}{d\cos\theta} = 2|\mathbf{p_1}||\mathbf{q_1}|, \quad (6)$$

where $\cos\theta$ is the angle between $\mathbf{p_1}$ and $\mathbf{q_1}$. $\cos\theta$ is a frame-dependent quantity, so we must be careful to define the frame (often, the centre of mass frame is used). $s = (p_1 + p_2)^2$ is usually considered to be a constant of the scattering:

[1]Note that in the massless limit $m_{1,2} \ll E_{1,2}$, $\mathcal{F} = 2s$, where $s = (p_1 + p_2)^2$.

the centre of mass energy. $u = (p_1 - q_2)^2$ is a dependent variable: it can be phrased in terms of $\cos\theta$ and $\sqrt{s}$, or in terms of s, t, m_i and m_i'. We also write

$$\frac{d^3 q_2}{2E_{q_2}} = d^4 q_2 \delta(q_2^2 - {m_2'}^2)\theta(q_2^0) \tag{7}$$

where $\theta(x)$ is the Heaviside theta function (i.e. $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$) and

$$\frac{d^3 q_1}{2E_{q_1}} = \frac{\mathbf{q_1}^2 d|\mathbf{q_1}| d\cos\theta d\phi}{2E_{q_1}} = \frac{1}{4|\mathbf{p_1}|} dE_{q_1} d\phi dt. \tag{8}$$

Then, performing the q_2 and ϕ integrals,

$$\frac{d\sigma}{dt} = \frac{1}{8\pi\mathcal{F}|\mathbf{p_1}|} \int dE_{q_1} |\mathcal{M}|^2 \delta(s - {m_2'}^2 + {m_1'}^2 - 2q_1.(p_1 + p_2)). \tag{9}$$

To get a simple expression, we now boost to the *centre of mass* frame, so that if $p_1^\mu = (\sqrt{\mathbf{p_1}^2 + m_1^2}, \mathbf{p_1})$ then $p_2^\mu = (\sqrt{\mathbf{p_1}^2 + m_2^2}, -\mathbf{p_1})$. Considering the Mandlestam variable $s = (\sqrt{\mathbf{p_1}^2 + m_1^2} + \sqrt{\mathbf{p_1}^2 + m_2^2})^2$,

$$\Rightarrow |\mathbf{p_1}| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \qquad \mathcal{F} = 2\lambda^{1/2}(s, m_1^2, m_2^2) \tag{10}$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Then

$$\left(\frac{d\sigma}{dt}\right) = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)}. \tag{11}$$

Decay Rates

Here we again omit the derivation, leaving interested students to peruse Peskin and Schroeder or the Standard Model course. The partial decay rate (or 'partial width') for $i \to f$ is:

$$\boxed{\Gamma_f = \frac{1}{2E_{p_i}} \int dp_f \, |\mathcal{M}|^2}. \tag{12}$$

We note that Γ_f is *not* Lorentz invariant, transforming as one over the energy ($1/E_{p_i}$) of the decaying particle i. It is however conventional to quote decay widths in the *rest frame of the decaying particle*, where then $E_{p_i} = m$, its mass. The *total decay rate* is $\Gamma = \sum_f \Gamma_f$, whereas the *branching ratio* for a final state f is $BR(i \to f) = \Gamma_f / \Gamma$. Putting in the correct units, we have lifetime

$$\tau = 6.58 \times 10^{-25} \frac{1 \text{ GeV}}{\Gamma} \text{ seconds.}$$