PART II: Classification of semi-simple Lie algebras.

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Preliminary comments:

These notes appeared as part of an earlier version of the Symmetries and Particle Physics course notes. The notation is slightly different from that of the preceding part of the present course. However, since there are no universal conventions this is something you will have to get used to!

As in the bulk of the notes, the material is more detailed than the lectures. This is intentional since the course is not long enough to present the material in depth. However, in an examination you would only be expected to reproduce the proofs of the various propositions, theorems and lemmas at the level that was explicitly presented in the lectures. Nevertheless, it would be very beneficial for you to understand the content of these notes.

1 The systematic classification of semi-simple Lie algebras.

1.1 Cartan's Classification of Simple Lie Algebras

The simple Lie algebras have been completely classified by Cartan. They fall into four infinite classes and five exceptional Lie algebras. The four infinite classes are the "classical algebras" associated with classical groups:

Classical Notation	*Rank*	*Cartan's Notation*
$su(n+1,\mathbb{C})$	$n \geq 1$	A_n
$so(2n+1,\mathbb{C})$	$n \geq 1$	B_n
$sp(2n,\mathbb{C})$	$n \geq 1$	C_n
$so(2n,\mathbb{C})$	$n > 2$	D_n

Here $su(n+1,\mathbb{C})$ denotes the complexification of the Lie algebra of $SU(n+1)$, i.e. it consists of complex linear combinations of the traceless hermitian $n+1 \times n+1$ matrices. $so(2n,\mathbb{C})$ and $so(2n+1,\mathbb{C})$ are defined analogously. The Lie algebra $sp(2n)$ is the complexification of the Lie algebra of the Lie group $Sp(2n)$.

There are also five exceptional lie algebras denoted G_2, F_4, E_6, E_7, E_8 which have dimension 14, 52, 78, 133 and 248 respectively.

The rank of the algebra is the dimension of a maximal commuting subalgebra.

1.2 Cartan Subalgebras

Definition 1. *Let L be a complex semisimple Lie algebra. A Cartan subalgebra H is a complex subspace of L such that*

i) If $h_1, h_2 \in H$ then $[h_1, h_2] = 0$

ii) For all $v \in L$, if $[v, h] = 0$ for all $h \in H$ then $v \in H$.

iii) For all $h \in H$, the operator $\mathrm{ad}(h)$ is diagonalizable.

The conditions (i) and (ii) imply that H is a maximal commuting subalgebra of L. It is straightforward to construct a subalgebra satisfying (i) and (ii), by induction, but it is non-trivial to satisfy (iii). It can be shown (but not here) that if L is a complex semi-simple Lie algebra, then L has a Cartan subalgebra. Cartan subalgebras are not unique; however it can be shown that if H_1 and H_2 are two Cartan subalgebras of a matrix Lie algebra $\mathcal{L}(G)$ then there exists some $g \in G$ such that $H_1 = gH_2g^{-1}$. Hence the dimension of all Cartan subalgebras is equal. The dimension of Cartan subalgebra H is called the rank r of L.

The diagonalizability condition (iii) together with (i) is sufficient to ensure that if $\{h_1, \ldots, h_r\}$ is a basis for H then $\mathrm{ad}(h_1), \ldots, \mathrm{ad}(h_r)$ can be simultaneously diagonalized.

Definition 2. *Suppose that L is a complexified semi-simple Lie algebra of rank n, and H is a Cartan subalgebra. Let $\{h_1, \dots, h_n\}$ be a basis for H. Then as the $\mathrm{ad}(h_i)$ can be simultaneously diagonalized, it follows that L can be decomposed as*

$$L = H \oplus \sum_{\underline{\alpha}} L_{\underline{\alpha}} \tag{1-1}$$

where the $\underline{\alpha} \neq 0$ are vectors in $\mathbb{R}^n$ with

$$L_{\underline{\alpha}} = \{v \in L : [h_i, v] = \alpha_i v\} \tag{1-2}$$

The vectors $\underline{\alpha} \neq 0$ are called roots and $L_{\underline{\alpha}} \neq 0$ is called the root space. Although 0 is not a root we will set $L_0 = H$ for convenience.

Lemma 1.

i) $L_{\underline{\alpha}} \perp L_{\underline{\beta}}$ *if* $\underline{\alpha} + \underline{\beta} \neq 0$.

ii) The restriction of the Killing form κ to H is non-degenerate.

iii) Suppose $\underline{\alpha}$, $\underline{\beta}$ are roots. If $\underline{\alpha} + \underline{\beta}$ is a root then $[L_{\underline{\alpha}}, L_{\underline{\beta}}] \subset L_{\underline{\alpha}+\underline{\beta}}$, if $\underline{\alpha} + \underline{\beta}$ is not a root then $[L_{\underline{\alpha}}, L_{\underline{\beta}}] = 0$.

iv) If $\underline{\alpha}$ is a root then so is $-\underline{\alpha}$.

Proof

i) Suppose $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $x_{\underline{\beta}} \in L_{\underline{\beta}}$ then

$$(\alpha_i + \beta_i)\kappa(x_{\underline{\alpha}}, x_{\underline{\beta}}) = \kappa([h_i, x_{\underline{\alpha}}], x_{\underline{\beta}}) + \kappa(x_{\underline{\alpha}}, [h_i, x_{\underline{\beta}}]) \tag{1-3}$$

But $\kappa([h_i, x_{\underline{\alpha}}], x_{\underline{\beta}}) + \kappa(x_{\underline{\alpha}}, [h_i, x_{\underline{\beta}}]) = 0$ by the associativity of κ.

Hence $(\alpha_i + \beta_i)\kappa(x_{\underline{\alpha}}, x_{\underline{\beta}}) = 0$. So if $\underline{\alpha} + \underline{\beta} \neq 0$ then $\kappa(x_{\underline{\alpha}}, x_{\underline{\beta}}) = 0$ for all $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $x_{\underline{\beta}} \in L_{\underline{\beta}}$.

ii) Suppose that κ restricted to H is degenerate. Then there exists some $v \in H$ such that $\kappa(v, h) = 0$ for all $h \in H$. And if $\underline{\alpha}$ is a root then by the reasoning used in (i) it follows that $\kappa(v, x_{\underline{\alpha}}) = 0$ for all $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$. So it follows that $\kappa(v, \ell) = 0$ for all $\ell \in L$, in contradiction with the fact that κ is non-degenerate on L. So κ restricted to H is non-degenerate. Hence the equation $\alpha_i = \kappa(h_i, u_{\underline{\alpha}})$ can be solved for unique $u_{\underline{\alpha}}$.

iii) Suppose that $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $x_{\underline{\beta}} \in L_{\underline{\beta}}$.

Then from the Jacobi identity

$$\begin{aligned} [h_i, [x_{\underline{\alpha}}, x_{\underline{\beta}}]] &= [[h_i, x_{\underline{\alpha}}], x_{\underline{\beta}}] + [x_{\underline{\alpha}}, [h_i, x_{\underline{\beta}}]] \\ &= \alpha_i [x_{\underline{\alpha}}, x_{\underline{\beta}}] + \beta_i [x_{\underline{\alpha}}, x_{\underline{\beta}}] \\ &= (\alpha_i + \beta_i)[x_{\underline{\alpha}}, x_{\underline{\beta}}] \end{aligned} \tag{1-4}$$

Hence if $\alpha + \beta$ is a root then this implies that $[x_{\underline{\alpha}}, x_{\underline{\beta}}] \in L_{\underline{\alpha}+\underline{\beta}}$. If, however, $\underline{\alpha} + \underline{\beta}$ is not a root then one must have $[x_{\underline{\alpha}}, x_{\underline{\beta}}] = 0$.

iv) Suppose $-\underline{\alpha}$ is not a root. Suppose $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$. Then if $\underline{\beta}$ is any root, then $\underline{\alpha} + \underline{\beta} \neq 0$. Then by (i) if $x_{\underline{\beta}} \in L_{\underline{\beta}}$ then $\kappa(x_{\underline{\alpha}}, x_{\underline{\beta}}) = 0$. Similarly, also by the reasoning in (i), $\kappa(x_{\underline{\alpha}}, h) = 0$ for all $h \in H$. This then implies that $x_{\underline{\alpha}} = 0$, so $L_{\underline{\alpha}} = 0$, a contradiction. ■

Corollary 1. *If $\underline{\alpha}$ is a root then $[L_{\underline{\alpha}}, L_{-\underline{\alpha}}] \subset H$.*

Proof From the reasoning used to prove (iii) in the above lemma, if $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $x_{-\underline{\alpha}} \in L_{-\underline{\alpha}}$ then $[h_i, [x_{\underline{\alpha}}, x_{-\underline{\alpha}}]] = 0$ which implies $[x_{\underline{\alpha}}, x_{-\underline{\alpha}}] \in H$. ■

Lemma 2. *If $\underline{\alpha}$ is a root then there exists a unique $y_{\underline{\alpha}} \in H$ such that $\alpha_i = \kappa(h_i, y_{\underline{\alpha}})$.*

Proof As the restriction of κ to H is non-degenerate, the equation $\alpha_i = \kappa(h_i, y_{\underline{\alpha}})$ can be solved uniquely for $y_{\underline{\alpha}} \in H$. Note that as $\underline{\alpha} \neq 0$, $y_{\underline{\alpha}} \neq 0$. ■

Corollary 2. *Suppose that $\underline{\alpha}$ is a root. If $X \in L_{\underline{\alpha}}$, $Y \in L_{-\underline{\alpha}}$ then $[X, Y] = \kappa(X, Y) y_{\underline{\alpha}}$.*

Proof From the above it follows that $[X, Y] \in H$. If $h = u^i h_i \in H$ then

$$\begin{aligned} \kappa([X,Y],h) &= \kappa(X,[Y,h]) \\ &= \kappa(X, u^i \alpha_i Y) \\ &= u^i \alpha_i \kappa(X,Y) \\ &= \kappa(y_{\underline{\alpha}}, h)\kappa(X,Y) \end{aligned} \tag{1-5}$$

Hence $\kappa([X,Y] - \kappa(X,Y)y_{\underline{\alpha}}, h) = 0$ for all $h \in H$. But κ restricted to H is non-degenerate, so $[X,Y] = \kappa(X,Y)y_{\underline{\alpha}}$ as required. ■

Lemma 3. *Suppose that $\underline{\alpha}$ is a root. There exists some $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $x_{-\underline{\alpha}} \in L_{-\underline{\alpha}}$ such that $y_{\underline{\alpha}} = [x_{\underline{\alpha}}, x_{-\underline{\alpha}}]$ and $\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) \neq 0$.*

Proof Pick some $x_{-\underline{\alpha}} \in L_{-\underline{\alpha}}$ with $x_{-\underline{\alpha}} \neq 0$.

Suppose that $\kappa(x_{-\underline{\alpha}}, x_{\underline{\alpha}}) = 0$ for all $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$. Then $L_{-\underline{\alpha}} \perp L_{\beta}$ for all roots β and $L_{-\underline{\alpha}} \perp H$. Hence $L_{-\underline{\alpha}} \perp L$. But κ is non-degenerate on L, so this implies $L_{-\underline{\alpha}} = 0$, a contradiction.

So there must exist some $x_{\underline{\alpha}} \in L_{\underline{\alpha}}$ with $\kappa(x_{-\underline{\alpha}}, x_{\underline{\alpha}}) \neq 0$. By rescaling, we can take $\kappa(x_{-\underline{\alpha}}, x_{\underline{\alpha}}) = 1$

Then by the corollary above, one finds $y_{\underline{\alpha}} = [x_{\underline{\alpha}}, x_{-\underline{\alpha}}]$.

Next, suppose that $\underline{\beta}$ is a root. Consider $W = \bigoplus_{j \in \mathbb{Z}} L_{\underline{\beta} + j\underline{\alpha}}$. By (iii) of Lemma 19 it follows that W is an invariant subspace of ad $x_{\pm\underline{\alpha}}$ and W is also an invariant subspace of ad $y_{\underline{\alpha}}$.

Then

$$\mathrm{Tr}_W \ (\mathrm{ad}\ y_{\underline{\alpha}}) = \mathrm{Tr}_W \ (\mathrm{ad}\ [x_{\underline{\alpha}}, x_{-\underline{\alpha}}]) = \mathrm{Tr}_W \ [\mathrm{ad}\ x_{\underline{\alpha}}, \mathrm{ad}\ x_{-\underline{\alpha}}] = 0 \tag{1-6}$$

where here the trace Tr_{W} denotes the trace restricted to the subspace W. As $y_{\underline{\alpha}} \in H$, set $y_{\underline{\alpha}} = y^i_{\underline{\alpha}} h_i$.

Then note that

$$\text{Tr}_{\text{W}} \ (\text{ad } y_{\underline{\alpha}}) = \bigoplus_{j \in \mathbb{Z}} y^i_{\underline{\alpha}} (\beta_i + j\alpha_i) \text{dim } L_{\underline{\beta}+j\underline{\alpha}} \tag{1-7}$$

We therefore obtain the equality

$$y^i_{\underline{\alpha}} \beta_i \text{dim } W + y^i_{\underline{\alpha}} \alpha_i \sum_{j \in \mathbb{Z}} j \text{dim } L_{\underline{\beta}+j\underline{\alpha}} = 0 \tag{1-8}$$

Note that

$$\begin{aligned} \kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) &= \kappa(y_{\underline{\alpha}}, [x_{\underline{\alpha}}, x_{-\underline{\alpha}}]) \\ &= \kappa[y_{\underline{\alpha}}, x_{\underline{\alpha}}], x_{-\underline{\alpha}} \\ &= \kappa(y^i_{\underline{\alpha}} \alpha_i x_{\underline{\alpha}}, x_{-\underline{\alpha}}) \\ &= y^i_{\underline{\alpha}} \alpha_i \kappa(x_{\underline{\alpha}}, x_{-\underline{\alpha}}) \\ &= y^i_{\underline{\alpha}} \alpha_i \end{aligned} \tag{1-9}$$

Hence

$$y^i_{\underline{\alpha}} \beta_i \text{dim } W + \kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) \sum_{j \in \mathbb{Z}} j \text{dim } L_{\underline{\beta}+j\underline{\alpha}} = 0 \tag{1-10}$$

If $\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) = 0$ then $y^i_{\underline{\alpha}} \beta_i = 0$ for all roots $\underline{\beta}$. This implies that $[y_{\underline{\alpha}}, X_{\underline{\beta}}] = 0$ for all $X_{\underline{\beta}} \in L_{\underline{\beta}}$. Hence $[y_{\underline{\alpha}}, \ell] = 0$ for all $\ell \in L$. It follows that ad $y_{\underline{\alpha}} = 0$. So

$$\kappa(y_{\underline{\alpha}}, \ell) = \text{Tr } (\text{ad } y_{\underline{\alpha}} \text{ad } \ell) = 0 \tag{1-11}$$

for all $\ell \in \ell$, where the trace is now taken over L. As κ is non-degenerate, this implies that $y_{\underline{\alpha}} = 0$, a contradiction.

Hence $\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) \neq 0$. ■

Proposition 1. *For each root $\underline{\alpha}$ there is an associated $SU(2)$ algebra*

Proof

Recall that we have obtained $y_{\underline{\alpha}} \in H$ and $x_{\pm\underline{\alpha}} \in L_{\pm\underline{\alpha}}$ satisfying $\alpha_i = \kappa(h_i, y_{\underline{\alpha}})$, $\kappa(x_{\underline{\alpha}}, x_{-\underline{\alpha}}) = 1$, $\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) \neq 0$ and

$$[x_{\underline{\alpha}}, x_{-\underline{\alpha}}] = y_{\underline{\alpha}} \tag{1-12}$$

Set $J_{\pm\underline{\alpha}} = \frac{1}{\sqrt{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}} x_{\pm\underline{\alpha}}$ and $h_{\underline{\alpha}} = \frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} y_{\underline{\alpha}}$.

Then it is straightforward to verify that

$$[h_{\underline{\alpha}}, J_{\pm\underline{\alpha}}] = J_{\pm\underline{\alpha}}, \qquad [J_{+\underline{\alpha}}, J_{-\underline{\alpha}}] = h_{\underline{\alpha}} \tag{1-13}$$

So $h_{\underline{\alpha}}$ and $J_{\pm\underline{\alpha}}$ satisfy the complexified $SU(2)$ algebra ($h_{\underline{\alpha}}$ is analogous to J_3 and $J_{\pm\underline{\alpha}}$ are analogous to the raising and lowering operators $J_\pm$.)

Note however that we are not assuming any hermiticity properties for $h_{\underline{\alpha}}$ or $J_{\pm\underline{\alpha}}$. ■

Proposition 2. *If $\underline{\alpha}$ is a root then $L_{\pm\underline{\alpha}}$ are both 1-dimensional and $k\underline{\alpha}$ is not a root for any $k \in \mathbb{C}$ unless $k = \pm 1$.*

Proof

Define

$$V = \bigoplus_{k \in \mathbb{C}} L_{k\underline{\alpha}} \tag{1-14}$$

(including $k = 0$).

Note that if $z \in L_{k\underline{\alpha}}$ then $J_{\pm\underline{\alpha}} \in L_{(k\pm1)\underline{\alpha}}$ and it is straightforward to check that

$$[h_{\underline{\alpha}}, z] = kz \tag{1-15}$$

Hence V is invariant under the adjoint action with respect to $h_{\underline{\alpha}}$, and $J_{\pm\underline{\alpha}}$; $h_{\underline{\alpha}}$ acts on $L_{k\underline{\alpha}}$ with $SU(2)$ weigh k.

Consider the adjoint action of the $SU(2)$ generators $J_{\pm\underline{\alpha}}$ and $h_{\underline{\alpha}}$ on V. This representation of $SU(2)$ is not necessarily irreducible.

However, taking some $z \in L_{k\underline{\alpha}}$ one constructs an irreducible represenation by acting on z with all possible powers of $J_{+\underline{\alpha}}$ and $J_{-\underline{\alpha}}$. Then by the reasoning used in the analysis of the $SU(2)$ irreducible representations, it follows that as ad $(h_{\underline{\alpha}})z = [h_{\underline{\alpha}}, z] = kz$ we must have $2k \in \mathbb{Z}$ (note that hermiticity properties of the $SU(2)$ generators are not required to obtain this result).

Consider the number of times that 0 appears as a $SU(2)$ weight. This must be n times as H has dimension n, we will identify where these zero weights appear in the decomposition of V into subspaces on which the $SU(2)$ action is irreducible.

First note that as $\kappa(h_{\underline{\alpha}}, h_{\underline{\alpha}}) \neq 0$ we can decompose H into a 1-dimensional subspace spanned by $h_{\underline{\alpha}}$ and a $n-1$-dimensional subspace spanned by elements of H which are orthogonal to $h_{\underline{\alpha}}$ with respect to the Killing form. Suppose $u = u^i h_i \in H$ with $\kappa(h_{\underline{\alpha}}, u) = 0$.

Then

$$[J_{\pm\underline{\alpha}}, u] \quad = \quad -[u^i h_i, J_{\pm\underline{\alpha}}]$$

$$\begin{aligned} &= -u^i[h_i, J_{\pm\underline{\alpha}}] \\ &= \mp u^i \alpha_i J_{\pm\underline{\alpha}} \\ &= \mp\kappa(u, y_{\underline{\alpha}}) J_{\pm\underline{\alpha}} \\ &= 0 \end{aligned} \tag{1-16}$$

because $\kappa(u, y_{\underline{\alpha}}) = 0$. Hence the $n-1$ linearly independent states in H which are orthogonal to $h_{\underline{\alpha}}$ are $SU(2)$ singlets with weight 0.

Also, observe that there is a $SU(2)$ triplet of weights $-1, 0, 1$ which corresponds from the adjoint action of $J_{\pm\underline{\alpha}}$ and $h_{\underline{\alpha}}$ on themselves.

This accounts for all occurrences of the weight 0.

Hence there are no other integral weights, for if there were then the corresponding multiplet would contain 0 as a weight, and this cannot be so, for we have accounted for all of the 0-weights.

Hence it follows that $L_{\underline{\alpha}}$ must have dimension 1. (If it had greater dimension then there would be too many linearly independent states of weight 0).

Also, $2\underline{\alpha}$ cannot be a root.

This then implies that $\frac{1}{2}\underline{\alpha}$ cannot be a root, for if it were, then both $\frac{1}{2}\underline{\alpha}$ and $2(\frac{1}{2}\underline{\alpha})$ would be roots, in contradiction with the above.

Hence $k = \frac{1}{2}$ cannot be a weight. This then removes all weights of the type $N + \frac{1}{2}$ for $N \in \mathbb{Z}$, because if such a weight were allowed, then $k = \frac{1}{2}$ would be an allowed weight.

It follows then that

$$V = H \oplus L_{\underline{\alpha}} \oplus L_{-\underline{\alpha}} \tag{1-17}$$

where $L_{\pm\underline{\alpha}}$ are 1-dimensional. ■.

Proposition 3. *Suppose $\underline{\alpha}$, $\underline{\beta}$ are roots. Consider the vectors $\underline{\beta} + n\underline{\alpha}$ for $n \in \mathbb{Z}$. This sequence consists of a string of roots for $p \leq n \leq q$ for some $p, q \in \mathbb{Z}$ with $p \leq 0 \leq q$. Moreover, $\frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} = -(p+q)$.*

Proof

Consider the space

$$V = \bigoplus_{n\in\mathbb{Z}} L_{\underline{\beta}+n\underline{\alpha}} \tag{1-18}$$

where the sum is taken over those n such that $\underline{\beta} + n\underline{\alpha}$ is a root. Then V is invariant under the adjoint action of the $SU(2)$ generators $h_{\underline{\alpha}}$, $J_{\pm\underline{\alpha}}$.

As each $L_{\underline{\beta}+n\underline{\alpha}}$ is one-dimensional, it follows that the representation is irreducible, the elements of V consist of elements of the form $(\text{ad } J_{\underline{\alpha}})^{m_1} J_{\underline{\beta}}$ and $(\text{ad } J_{-\underline{\alpha}})^{m_2} J_{\underline{\beta}}$ for $m_1, m_2 \in \mathbb{N}$.

Hence there exist integers p, q with $p \leq 0 \leq q$ such that $\underline{\beta} + n\underline{\alpha}$ is a root if and only if $p \leq n \leq q$.

Note that if $X \in L_{\underline{\beta}+n\underline{\alpha}}$ then

$$\begin{aligned}
[h_{\underline{\alpha}}, X] &= \frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}[y_{\underline{\alpha}}, X] \\
&= \frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}(y^i_{\underline{\alpha}})[h_i, X] \\
&= \frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}(y^i_{\underline{\alpha}})(\beta_i + n\alpha_i)X \\
&= (n + \frac{\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})})X
\end{aligned} \tag{1-19}$$

But the largest and smallest of the possible eigenvalues is $\pm r$ where $2r \in \mathbb{N}$, i.e.

$$q + \frac{\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} = r, \qquad p + \frac{\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} = -r \tag{1-20}$$

and hence

$$\frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} = -(p + q) \tag{1-21}$$

as required. ■.

Definition 3. *Suppose $\underline{\alpha}$, $\underline{\beta}$ are roots. The roots $\underline{\beta} + n\underline{\alpha}$ for $n \in \mathbb{Z}$. where $p \leq n \leq q$ for some $p, q \in \mathbb{Z}$ with $p \leq 0 \leq q$ are called the $\underline{\alpha}$-string of roots through $\underline{\beta}$.*

Corollary 3. *If $\underline{\alpha}$ and $\underline{\beta}$ are roots then so is $\underline{\beta} - \frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}\underline{\alpha}$*

Proof Consider the string of roots $\underline{\beta} + n\underline{\alpha}$, $n \in \mathbb{Z}$, $p \leq n \leq q$ for some $p, q \in \mathbb{Z}$ with $p \leq 0 \leq q$ passing through $\underline{\beta}$. We have shown that $\frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} = -(p + q)$

Then it follows that $p \leq -\frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} \leq q$.

Hence $\underline{\beta} - \frac{2\kappa(y_{\underline{\beta}}, y_{\underline{\alpha}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})}\underline{\alpha}$ is a root. ■

Lemma 4. *Suppose $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\alpha} + \underline{\beta}$ are roots. Then $[L_{\underline{\alpha}}, L_{\underline{\beta}}] = L_{\underline{\alpha}+\underline{\beta}}$.*

Proof The adjoint action of $J_{\pm\underline{\alpha}}$ and $h_{\underline{\alpha}}$ on $V = \bigoplus_{n\in\mathbb{Z}} L_{\underline{\beta}+n\underline{\alpha}}$ is irreducible, as all the root spaces are 1-dimensional.

Consider the string of roots containing $\underline{\beta}$ obtained by acting on $J_{\underline{\beta}}$ with $J_{\underline{\alpha}}$, corresponding to an irreducible representation of $SU(2)$ on V. $\underline{\beta} + \underline{\alpha}$ lies in this string, hence $[J_{\underline{\alpha}}, J_{\underline{\beta}}] \neq 0$. As $[J_{\underline{\alpha}}, J_{\underline{\beta}}] \in L_{\underline{\alpha}+\underline{\beta}}$ and all the root spaces are 1-dimensional, it follows that $[L_{\underline{\alpha}}, L_{\underline{\beta}}] = L_{\underline{\alpha}+\underline{\beta}}$. ■

Proposition 4. *Suppose that L is a complex semisimple Lie algebra. Then H is spanned by the $y_{\underline{\alpha}}$.*

Proof

Suppose that the $y_{\underline{\alpha}}$ do not span H. Then there exists some non-vanishing $y_\perp \in H$ which is orthogonal to all the $y_{\underline{\alpha}}$. It follows that $[J_{\underline{\alpha}}, y_\perp] = 0$ for all roots $\underline{\alpha}$, and so $y_\perp$ commutes with all elements of L. This in turn implies ad $y_\perp = 0$.

Then $\kappa(y_\perp, \ell) = \text{Tr}\,(\text{ad}\ y_\perp \text{ad}\ \ell) = 0$ for all $\ell \in L$. As κ is non-degenerate on L, it follows that $y_\perp = 0$, a contradiction. ■

Proposition 5. *There exists a basis of H with respect to which the components of the $y_{\underline{\alpha}}$ and of the restriction of the Killing form to H are real. In addition, κ is positive definite over the span of the $y_{\underline{\alpha}}$ over $\mathbb{R}$.*

Proof

Consider the $y_{\underline{\alpha}}$ for all roots $\underline{\alpha}$. These span H and are all non-vanishing. Hence, there exists a basis S of H consisting of some subset of the $y_{\underline{\alpha}}$.

In this basis, consider $y_{\underline{\beta}}$ for roots $\underline{\beta}$. We have shown that the ad $y_{\underline{\beta}}$ are simultaneously diagonalizable over L with real eigenvalues.

Suppose that $\lambda_{\underline{\alpha}}$ are some real constants. Then ad $(\sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} y_{\underline{\alpha}})$ is diagonal with real diagonal entries, so on taking the trace we find that $\kappa(\sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} y_{\underline{\alpha}}, \sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} y_{\underline{\alpha}})$ is a sum of squares of reals, and hence is non-negative, and can only vanish if ad $\sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} y_{\underline{\alpha}} = 0$, which in turn implies that $\sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} y_{\underline{\alpha}} = 0$ as the Killing form is non-degenerate on a semi-simple Lie algebra.

This implies that $\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}}) > 0$, and κ is positive definite on the real span of the $y_{\underline{\alpha}}$.

If $\underline{\alpha}, \underline{\beta}$ are any two roots, recall that

$$\frac{2\kappa(y_{\underline{\alpha}}, y_{\underline{\beta}})}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} \in \mathbb{Z} \tag{1-22}$$

Hence it follows that $\kappa(y_{\underline{\alpha}}, y_{\underline{\beta}}) \in \mathbb{R}$, so the components of κ restricted to H are real in the basis S.

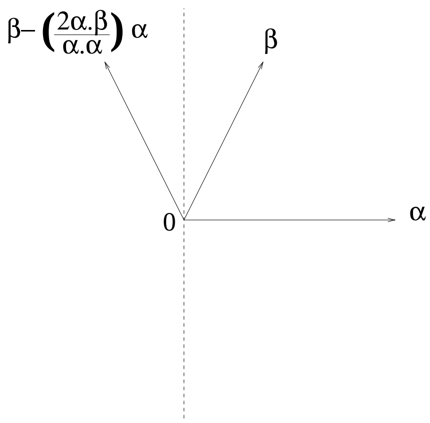
Therefore, one can construct an orthonormal (with respect to κ) basis S' of H consisting of real linear combinations of elements of S. It follows that if y_i is a basis element of S' and $\underline{\alpha}$ is a root then $\kappa(y_i, y_{\underline{\alpha}}) \in \mathbb{R}$. ■

1.3 Geometric Properties of Roots

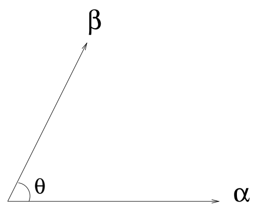
We have shown that there exists a basis of H, $S' = \{y_i, i = 1, \dots, n\}$ such that $\kappa(y_i, y_j) = \delta_{ij}$, and $\kappa_(y_i, y_{\underline{\alpha}}) \in \mathbb{R}$ for all roots $\underline{\alpha}$. In this basis, $\alpha_i = \kappa(y_i, y_{\underline{\alpha}})$ so the components of the root $\underline{\alpha}$ are also real. One can view the root space as being isomorphic to $\mathbb{R}^n$. Note that as the

$y_{\underline{\alpha}}$ span H it follows that the $\underline{\alpha}$ span $\mathbb{R}^n$. Henceforth we shall adopt this basis on H, so one can consider the roots as points in $\mathbb{R}^n$ equipped with the standard inner product.

Recall that if $\underline{\alpha}$ and $\underline{\beta}$ are roots then so is $\underline{\beta} - \left(\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}}\right)\underline{\alpha}$. This has a simple interpretation in terms of reflection symmetry; because $\underline{\beta} - \left(\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}}\right)\underline{\alpha}$ is the reflection of $\underline{\beta}$ in the hyperplane passing through the origin with normal parallel to $\underline{\alpha}$.



These reflection symmetries generate a group which is called the Weyl group. One can also obtain further constraints on possible roots. In particular, suppose that $\underline{\alpha}$ and $\underline{\beta}$ ($\underline{\beta} \neq \pm\underline{\alpha}$) are roots with angle θ subtending between them.



Then

$$\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}} = 2\frac{|\underline{\beta}|}{|\underline{\alpha}|}\cos\theta \in \mathbb{Z} \tag{1-23}$$

and

$$\frac{2\underline{\beta}.\underline{\alpha}}{\underline{\beta}.\underline{\beta}} = 2\frac{|\underline{\alpha}|}{|\underline{\beta}|}\cos\theta \in \mathbb{Z} \tag{1-24}$$

Taking the product we find that $4\cos^2\theta \in \mathbb{Z}$. So $4\cos^2\theta$ must be an non-negative integer ≤ 4

This condition means that θ must be an integer multiple of $\frac{\pi}{4}$ or $\frac{\pi}{6}$.

Moreover, given such θ, if $|\underline{\beta}| \geq |\underline{\alpha}|$ then the constraint $2\frac{|\underline{\alpha}|}{|\underline{\beta}|}\cos\theta \in \mathbb{Z}$ imposes strict constraints on possible values of the ratio $\frac{|\underline{\alpha}|}{|\underline{\beta}|}$. The possibilities can be tabulated straightforwardly

$\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}}$	$\frac{2\underline{\beta}.\underline{\alpha}}{\underline{\beta}.\underline{\beta}}$	θ	$\frac{\lvert\underline{\beta}\rvert}{\lvert\underline{\alpha}\rvert}$
0	0	$\frac{\pi}{2}$	Undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
2	1	$\frac{\pi}{4}$	$\sqrt{2}$
-2	-1	$\frac{3\pi}{4}$	$\sqrt{2}$
3	1	$\frac{\pi}{6}$	$\sqrt{3}$
-3	-1	$\frac{5\pi}{6}$	$\sqrt{3}$

Moreover, given a root $\underline{\beta}$ there is an $\underline{\alpha}$-string passing through $\underline{\beta}$, of the form $\underline{\beta} + n\underline{\alpha}$ for $p \leq n \leq q$. Suppose that $\underline{\beta}$ lies at the end of the string. Then either $p = 0$ or $q = 0$.

Now we have shown that

$$\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}} = -(p+q) \tag{1-25}$$

and there are $q - p + 1$ roots in the string. Hence it follows that there are $|\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}}| + 1$ roots in the string. But from the tabulation of possible values of $\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}}$ it follows that the number of possible roots on the string cannot exceed 4.

1.4 Simple Roots

Although the symmetries derived so far place considerable constraints on possible roots, it will be convenient to single out a special class of roots, called simple roots. To do this, we must first define an ordering on the space of roots.

Definition 4. *Choose some $\underline{p} \in \mathbb{R}^n$ such that $\kappa^{ij}\underline{p}_i\underline{\alpha}_j \neq 0$ for all roots $\underline{\alpha}$. Then one obtains an ordering on vectors $\underline{\beta}$, $\underline{\chi}$ in $\mathbb{R}^n$ by defining $\underline{\beta} > \underline{\chi}$ if $\kappa^{ij}\underline{p}_i\underline{\beta}_j > \kappa^{ij}\underline{p}_i\underline{\chi}_j$. In particular, $\underline{\beta} \in \mathbb{R}^n$ is said to be positive if $\kappa^{ij}\underline{p}_i\underline{\beta}_j > 0$.*

Geometrically, if one works with a basis in which $\kappa_{ij} = \delta_{ij}$, then it is clear that $\underline{p}$ can be found. This is because there are only finitely many roots, hence there exists some hyperplane through the origin which contains no roots. Then one can take $\underline{p}$ to be a normal vector to this hyperplane. With this definition, it is clear that a root must be either positive or negative; and the sum of two positive roots is also positive. The hyperplane so chosen splits the root space into two halves- the positive and negative roots.

There is clearly some ambiguity in the choice of $\underline{p}$; though it makes no difference to the classification of semisimple Lie algebras and their representations.

Definition 5. *A root is simple if it is positive and cannot be written as a sum of positive roots.*

Lemma 5. *If $\underline{\alpha}$ and $\underline{\beta}$ are different simple roots, then $\underline{\alpha} - \underline{\beta}$ is not a root.*

Proof Suppose that $\underline{\alpha} - \underline{\beta}$ is a root. If $\underline{\beta} > \underline{\alpha}$ then $\underline{\beta} - \underline{\alpha}$ is positive and one can write $\underline{\beta} = (\underline{\beta} - \underline{\alpha}) + \underline{\alpha}$ which is a sum of two positive roots, which is a contradiction as $\underline{\beta}$ is simple. Similarly, if $\underline{\alpha} > \underline{\beta}$ then $\underline{\alpha} - \underline{\beta}$ is positive and $\underline{\alpha} = (\underline{\alpha} - \underline{\beta}) + \underline{\beta}$, again a contradiction. So $\underline{\alpha} - \underline{\beta}$ cannot be a root. ■

Lemma 6. *If $\underline{\alpha}$ and $\underline{\beta}$ are different simple roots then $\underline{\alpha}.\underline{\beta} \leq 0$*

Proof Consider the $\underline{\alpha}$ string of roots through $\underline{\beta}$, $\underline{\beta} + n\underline{\alpha}$ for $p \leq n \leq q$ $(p \leq 0 \leq q)$.

Then $\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}} = -(p+q)$. But $\underline{\beta} - \underline{\alpha}$ is not a root, hence we must have $p = 0$, so $\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\alpha}.\underline{\alpha}} = -q \leq 0$. ■

Lemma 7. *The set of simple roots is linearly independent over $\mathbb{R}$.*

Proof

Suppose that $\underline{\alpha}_i$ are simple roots, and set $\underline{\gamma} = \sum_i c_i \underline{\alpha}_i$ where c_i are real constants. Suppose that $\underline{\gamma} = 0$. Split $\underline{\gamma}$ into two parts

$$\underline{\gamma} = \underline{\mu} - \underline{\nu} \tag{1-26}$$

where

$$\underline{\mu} = \sum_{c_i > 0} c_i \underline{\alpha}_i, \qquad \underline{\nu} = -\sum_{c_i < 0} c_i \underline{\alpha}_i \tag{1-27}$$

By construction $\underline{\mu}$ and $\underline{\nu}$ are positive vectors. Then

$$(\underline{\mu} - \underline{\nu}).(\underline{\mu} - \underline{\nu}) = \underline{\mu}.\underline{\mu} + \underline{\nu}.\underline{\nu} - 2\underline{\mu}.\underline{\nu} \geq \underline{\mu}.\underline{\mu} + \underline{\nu}.\underline{\nu} > 0 \tag{1-28}$$

where we have used the fact that $\underline{\mu}.\underline{\nu} \leq 0$ using the result of the previous lemma. ■

Lemma 8. *If $\underline{\alpha}$ is a positive root then it can be written as a linear combination*

$$\underline{\alpha} = \sum_i c_i \underline{\beta}_i \tag{1-29}$$

where $c_i \in \mathbb{N}$ and $\underline{\beta}_i$ are simple roots.

Proof

If $\underline{\alpha}$ is simple then we are done. Otherwise, it is not, and can be written as a sum of two positive roots. By repeating this process (which must end after a finite number of steps), one must eventually obtain the decomposition into simple roots. ■

Lemma 9. *There are n simple roots (where L is of rank n).*

Proof

As the simple roots are linearly independent, there cannot be more than n of them.

Suppose that there are fewer than n simple roots. Then there exists some non-vanishing $\underline{\omega} \in \mathbb{R}^n$ which is orthogonal to all simple roots $\underline{\alpha}$. As all positive (and hence negative) roots can be written as a linear combination of simple roots, it follows that $\underline{\omega}$ is orthogonal to all roots $\underline{\beta}$. This implies that there exists some $w \in H$ which commutes with all root spaces $L_{\underline{\alpha}}$, in contradiction with the non-degeneracy of the Killing form.

Hence there must be n simple roots. ■.

Next we shall show that by generating all possible root strings from the simple roots, one obtains the whole root space in a unique fashion.

Lemma 10. *Suppose that $\underline{\alpha}$ is a positive, but non-simple root. There exists some simple root $\underline{\beta}$ such that $\underline{\alpha} - \underline{\beta}$ is a root.*

Proof

Suppose that $\underline{\alpha}$ is positive, and that $\underline{\alpha} - \underline{\beta}$ is not a root for all simple roots $\underline{\beta}$.

Now suppose that $\underline{\beta}$ is a simple root.

If $\underline{\alpha}.\underline{\beta} > 0$ then the $\underline{\beta}$ string through $\underline{\alpha}$, $\underline{\alpha} + n\underline{\beta}$, $p \leq n \leq q$ must have $p < 0$. Hence $\underline{\alpha} - \underline{\beta}$ must be a root. Contradiction.

Hence we must have $\underline{\alpha}.\underline{\beta} \leq 0$ for all simple roots. But as $\underline{\alpha}$ can be written as a (non-negative) integer linear combination of simple roots this implies $\underline{\alpha}.\underline{\alpha} \leq 0$, a contradiction. ■

Definition 6. *Suppose that $\underline{\alpha}$ is a positive root. Then $\underline{\alpha}$ can be written uniquely as a sum $\underline{\alpha} = \sum_i c_i \underline{\alpha}_i$ where $c_i \in \mathbb{N}$ and the $\underline{\alpha}_i$ are simple.*

We define $c = \sum_i c_i$ to be the height of the root $\underline{\alpha}$.

Lemma 11. *Suppose that $\underline{\alpha}$ is a positive root. Then one can write $\underline{\alpha} = \underline{\alpha}_1 + \cdots + \underline{\alpha}_k$ where the $\underline{\alpha}_i$ are simple roots which are not necessarily distinct and the partial sums $\underline{\alpha}_1 + \cdots + \underline{\alpha}_i$ are roots*

Proof

We will prove the lemma by induction on the height of $\underline{\alpha}$.

First note that it is true for roots of height 1, as these are just simple roots. Suppose that it is true for roots of height $1 \leq m$.

Then let $\underline{\alpha}$ be a root of height $m + 1$. As $\underline{\alpha}$ is not simple, there exists some simple $\underline{\beta}$ such that $\underline{\chi} = \underline{\alpha} - \underline{\beta}$ is a root. Then $\underline{\beta} = \underline{\alpha} - \underline{g}$. $\underline{g}$ cannot be negative, for then $-\underline{g}$ would be positive, and one could write $\underline{\beta}$ as a sum of two positive roots, which is a contradiction because $\underline{\beta}$ is simple. Hence $\underline{g}$ must be positive.

Then $\underline{\alpha} = \underline{\chi} + \underline{\beta}$. $\underline{\chi}$ has height $\leq m$, and can be decomposed as $\underline{\chi} = \underline{\beta}_1 + \dots \underline{\beta}_k$ for simple $\underline{\beta}_i$ such that the partial sums $\underline{\beta}_1 + \dots + \underline{\beta}_i$ are roots. Setting $\underline{\beta}_{k+1} = \underline{\beta}$ we therefore obtain the same decomposition for $\underline{\alpha}$, $\underline{\alpha} = \underline{\beta}_1 + \dots + \underline{\beta}_k + \underline{\beta}_{k+1}$. ■

Lemma 12. *Suppose that $\underline{\alpha}$ is a positive non-simple root and $\underline{\beta}$ is simple. Then the $\underline{\beta}$-string of roots passing through $\underline{\alpha}$ consists entirely of positive roots.*

Proof

Suppose $\underline{\chi} = \underline{\alpha} + n\underline{\beta}$ for $n \in \mathbb{Z}$, lies on the root string. If $n \geq 0$ then clearly $\underline{\chi}$ must be positive.

Suppose $n < 0$. If $\underline{\chi}$ is negative then there exists some $m \in \mathbb{Z}$, $n < m \leq 0$ such that $\underline{g} \equiv \underline{\alpha} + m\underline{\beta}$ is a positive root but $\underline{f} \equiv \underline{\alpha} + (m-1)\underline{\beta}$ is a negative root (such m must exist as we know that $\underline{\alpha}$ is positive but $\underline{\chi}$ is negative).

But $\underline{g} - \underline{f} = \underline{\beta}$, which implies that $\underline{\beta}$ can be written as a sum of two positive roots, a contradiction, as $\underline{\beta}$ is simple. ■.

Bringing these results together we have

Theorem 1. *Any positive root can be obtained by proceeding from a simple root along root strings generated by simple roots. The root space structure is determined uniquely by the simple roots.*

Proof

We have shown that if $\underline{\alpha}$ is a positive root, then $\underline{\alpha} = \underline{\alpha}_1 + \dots + \underline{\alpha}_k$ where the $\underline{\alpha}_i$ are simple roots which are not necessarily distinct and the partial sums $\underline{\alpha}_1 + \dots + \underline{\alpha}_i$ are roots.

Hence one can obtain $\underline{\alpha}$ by starting from $\underline{\alpha}_1$, then passing to $\underline{\alpha}_1 + \underline{\alpha}_2$ along the $\underline{\alpha}_2$ string through $\underline{\alpha}_1$, then to $\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3$ along the $\underline{\alpha}_3$ string through $\underline{\alpha}_1 + \underline{\alpha}_2$ and similarly until one obtains $\underline{\alpha}$.

To establish that this construction determines the whole root space structure, observe that the roots of height $k+1$ are determined by the roots of height $\leq k$. To see this, note that all roots of height $k+1$ can be obtained by passing along root strings generated by simple roots. However, all strings passing through roots of height k are determined uniquely by an end-point to the string of height $\leq k$ (as knowing where one end-point is enables one to compute the string length and hence the other end-point.)

Once the positive roots $\underline{\alpha}$ have been fixed in this fashion, the negative roots $\underline{\beta}$ are given by $\underline{\beta} = -\underline{\alpha}$. ■

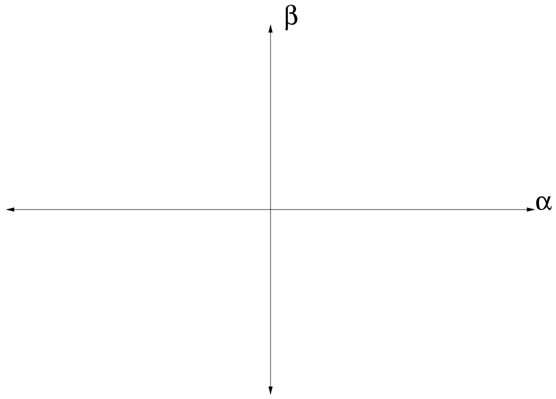
1.5 Rank Two Examples

It is instructive to compute all the possible rank 2 root diagrams, and plot them on the plane. In these cases, there are exactly two simple roots $\underline{\alpha}$, $\underline{\beta}$ and without loss of generality we can take $|\underline{\beta}| \geq |\underline{\alpha}|$. As we require that $\underline{\alpha}.\underline{\beta} \leq 0$ there are only four possible angles between the simple roots; $\theta = \frac{\pi}{2}$, $\theta = \frac{2\pi}{3}$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{6}$. Our strategy will be to first obtain all the positive roots which can be obtained by passing along simple root strings. This then fixes all the negative roots as well.

1.5.1 Simple roots with $\theta = \frac{\pi}{2}$

In this case $\underline{\alpha}.\underline{\beta} = 0$ so the $\underline{\beta}$ string passing through $\underline{\alpha}$ has only one element, and the $\underline{\alpha}$ string passing through $\underline{\beta}$ also has only one element. The only positive roots are therefore $\underline{\alpha}$ and $\underline{\beta}$.

Therefore, there are four roots $\pm\underline{\alpha}$ and $\pm\underline{\beta}$ with $\underline{\alpha}$ and $\underline{\beta}$ orthogonal, this corresponds to the (non-simple) Lie algebra $A_1 \oplus A_1$:



The Lie algebra associated with this root diagram is not simple. This follows from the proposition:

Proposition 6. *Suppose that the root space of a semi-simple Lie algebra L decomposes into $R \cup R_\perp$ with roots in R orthogonal to those in $R_\perp$. Then L is not simple.*

Proof

First we show that $R \cap R_\perp = \emptyset$. Suppose that $\underline{\alpha} \in R \cap R_\perp$. Then $\underline{\alpha}$ is orthogonal to all roots. Hence $h_{\underline{\alpha}}$ must commute with all elements the root spaces, and thus with the whole of L. Hence ad $h_{\underline{\alpha}} = 0$, which implies $\kappa(h_{\underline{\alpha}}, \ell) = 0$ for all $\ell \in L$. As κ is non-degenerate, this implies $h_{\underline{\alpha}} = 0$, a contradiction.

Let $\underline{\alpha}$ denote the roots in R, and $\underline{\tilde{\alpha}}$ denote roots in $R_\perp$, with corresponding elements $h_{\underline{\alpha}}$ and $h_{\underline{\tilde{\alpha}}}$ of H. Denote by L_1 the span of the $h_{\underline{\alpha}}$ and $J_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and by L_2 the span of the $h_{\underline{\tilde{\alpha}}}$ and $J_{\underline{\tilde{\alpha}}} \in L_{\underline{\tilde{\alpha}}}$. Then L_1 is a Lie subalgebra of L because if $\underline{\alpha} \in R$, $[h_{\underline{\alpha}}, h_{\underline{\beta}}] = 0$, $[h_{\underline{\alpha}}, J_{\underline{\beta}}] \in L_{\underline{\beta}}$ and if $\underline{\alpha}$, $\underline{\beta} \in R$ and $\underline{\alpha} + \underline{\beta}$ is a root then $\underline{\alpha} + \underline{\beta} \in R$ and $[J_{\underline{\alpha}}, J_{\underline{\beta}}] \in L_{\underline{\alpha}+\underline{\beta}}$, and if $\underline{\alpha} + \underline{\beta}$ is not a root then $[J_{\underline{\alpha}}, J_{\underline{\beta}}] = 0$. Similarly L_2 is a Lie subalgebra of L.

Next we prove that L_1 is an ideal of L. To do this, it suffices to prove that $[\ell, h_{\underline{\alpha}}] \in L_1$ and $[\ell, J_{\underline{\alpha}}] \in L_1$ for all $\ell \in L$ and $\underline{\alpha} \in R$.

Note that if $\ell \in H$ then $[\ell, h_{\underline{\alpha}}] = 0 \in L_1$ and $[\ell, J_{\underline{\alpha}}] \in J_{\underline{\alpha}} \subset L_1$.

We have shown that if $\ell = J_{\underline{\beta}} \in R$ then $[\ell, h_{\underline{\alpha}}] \in L_1$ and $[\ell, J_{\underline{\alpha}}] \in L_1$.

The only remaining case is when $\ell = J_{\underline{\tilde{\beta}}}$ for some $\underline{\tilde{\beta}} \in R_\perp$. Then $[\ell, h_{\underline{\alpha}}] \in L_{\underline{\tilde{\beta}}}$.

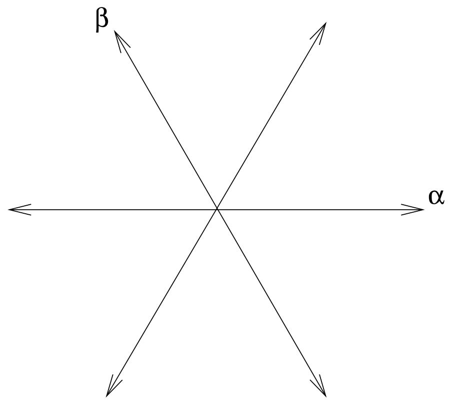
Lastly, consider $[J_{\underline{\tilde{\beta}}}, J_{\underline{\alpha}}]$. Note that $\underline{\alpha} + \underline{\tilde{\beta}}$ cannot be a root, because if it were, it would lie in either R or $R_\perp$. In either case, one would find $J_{\underline{\tilde{\beta}}} \in R \cap R_\perp$ or $J_{\underline{\alpha}} \in R \cap R_\perp$, a contradiction. Hence $[J_{\underline{\tilde{\beta}}}, J_{\underline{\alpha}}] = 0$. ■.

1.5.2 Simple roots with $\theta = \frac{2\pi}{3}$

In this case, $\frac{2\underline{\beta}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = \frac{2\underline{\alpha}.\underline{\beta}}{\underline{\beta}.\underline{\beta}} = -1$, It follows that the $\underline{\alpha}$ string through $\underline{\beta}$ has two elements, $\underline{\beta}$ and $\underline{\alpha} + \underline{\beta}$. The $\underline{\beta}$ string through $\underline{\alpha}$ also has two elements $\underline{\alpha}$ and $\underline{\alpha} + \underline{\beta}$.

Consider now simple root strings passing through $\underline{\alpha}+\underline{\beta}$. The $\underline{\beta}$ simple root string starts as $\underline{\alpha}$, and has therefore only the two roots already found ($\underline{\alpha}$ and $\underline{\alpha} + \underline{\beta}$). Similarly, the $\underline{\alpha}$ simple root string starts as $\underline{\beta}$, and has therefore only the two roots already found ($\underline{\beta}$ and $\underline{\alpha}+\underline{\beta}$) Hence it is not possible to obtain any additional positive roots by constructing simple root strings through $\underline{\alpha} + \underline{\beta}$.

So there are six roots $\pm\underline{\alpha}$, $\pm\underline{\beta}$ and $\pm(\underline{\alpha} + \underline{\beta})$. The root diagram is that of A_2, corresponding to the weight diagram of the adjoint representation of $SU(3)$.



All the roots in this diagram have the same length.

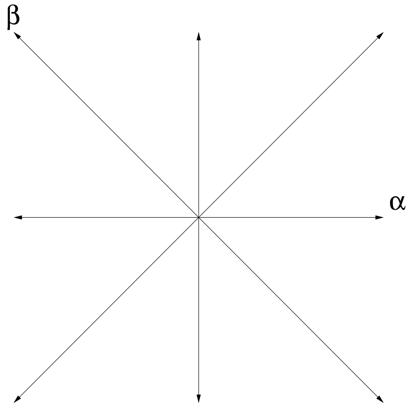
1.5.3 Simple roots with $\theta = \frac{3\pi}{4}$

In this case, $\frac{2\underline{\beta}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = -2$, $\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\beta}.\underline{\beta}} = -1$

Hence the $\underline{\alpha}$ root string through $\underline{\beta}$ has three roots $\underline{\beta}$, $\underline{\beta} + \underline{\alpha}$ and $\underline{\beta} + 2\underline{\alpha}$, and the $\underline{\beta}$ root string through $\underline{\alpha}$ has two roots, $\underline{\alpha}$ and $\underline{\alpha} + \underline{\beta}$.

Consider constructing simple root strings through $\underline{\beta} + n\underline{\alpha}$ for $n = 1, 2$. The $\underline{\alpha}$ root string has already been constructed. Consider the $\underline{\beta}$ root string. The $\underline{\beta}$ root string passing through $\underline{\alpha} + \underline{\beta}$ has also already been constructed. It remains to consider the $\underline{\beta}$ root string through $\underline{\beta} + 2\underline{\alpha}$; this contains only $\underline{\beta} + 2\underline{\alpha}$ as neither $2\underline{\alpha}$ or $2(\underline{\beta} + \underline{\alpha})$ can be roots.

So there are eight roots $\pm\underline{\alpha}$, $\pm\underline{\beta}$, $\pm(\underline{\alpha} + \underline{\beta})$ and $\pm(\underline{\beta} + 2\underline{\alpha})$, which give the root diagram of B_2:



The ratio of the long root length to short root length is $\sqrt{2}$.

1.5.4 Simple roots with $\theta = \frac{5\pi}{6}$

In this case, $\frac{2\underline{\beta}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = -3$, $\frac{2\underline{\alpha}.\underline{\beta}}{\underline{\beta}.\underline{\beta}} = -1$,

Hence the $\underline{\alpha}$ root string through $\underline{\beta}$ contains 4 roots, $\underline{\beta}$, $\underline{\beta} + \underline{\alpha}$, $\underline{\beta} + 2\underline{\alpha}$ and $\underline{\beta} + 3\underline{\alpha}$. The $\underline{\beta}$ root string through $\underline{\alpha}$ contains only two roots, $\underline{\alpha}$ and $\underline{\alpha} + \underline{\beta}$.

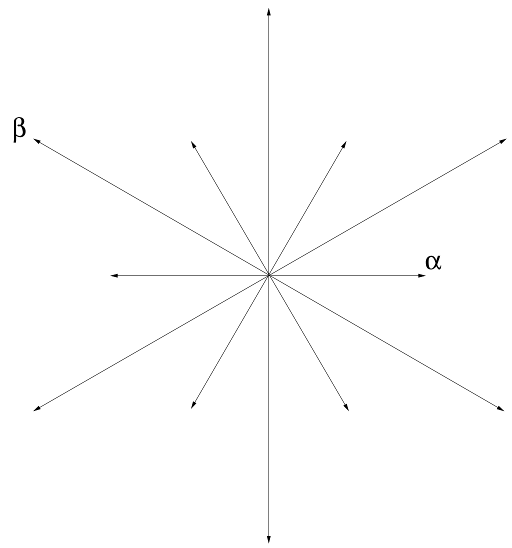
Consider then simple root strings through $\underline{\beta} + n\underline{\alpha}$ for $n = 1, 2, 3$. As we have already constructed the $\underline{\alpha}$ root string through these points, it suffices to consider the $\underline{\beta}$ root string. The $\underline{\beta}$ root string through $\underline{\beta} + \underline{\alpha}$ has already been constructed. The $\underline{\beta}$ root string through $\underline{\beta} + 2\underline{\alpha}$ can only contain $\underline{\beta} + 2\underline{\alpha}$ as neither $2\underline{\alpha}$ or $2(\underline{\beta} + \underline{\alpha})$ can be roots.

Consider the $\underline{\beta}$ root string through $\underline{\beta} + 3\underline{\alpha}$. This cannot contain $3\underline{\alpha}$, so $\underline{\beta} + 3\underline{\alpha}$ is an endpoint of this string. On computing $-2\frac{(\underline{\beta}+3\underline{\alpha}).\underline{\beta}}{\underline{\beta}.\underline{\beta}} = 1$ we see that this string must contain exactly two elements, $\underline{\beta} + 3\underline{\alpha}$ and $2\underline{\beta} + 3\underline{\alpha}$.

Lastly, consider a $\underline{\alpha}$ string through $2\underline{\beta} + 3\underline{\alpha}$. As $\underline{\alpha}.(2\underline{\beta} + 3\underline{\alpha}) = 0$ it follows that this string contains just $2\underline{\beta} + 3\underline{\alpha}$.

Hence there are 12 roots, $\pm\underline{\alpha}$, $\pm\underline{\beta}$, $\pm(3\underline{\alpha} + 2\underline{\beta})$, $\pm(\underline{\alpha} + \underline{\beta})$, $\pm(2\underline{\alpha} + \underline{\beta})$, $\pm(3\underline{\alpha} + \underline{\beta})$.

This root diagram corresponds to the exceptional group G_2.



The ratio of the long root length to short root length is $\sqrt{3}$.

1.6 Dynkin Diagrams

Definition 7. *Suppose that* $\underline{\alpha}_i$ *are the simple roots associated with a complex semi-simple Lie algebra. Then the Cartan matrix* K *is a matrix of integers with components given by*

$$K_{ij} = \frac{2\underline{\alpha}_i.\underline{\alpha}_j}{\underline{\alpha}_j.\underline{\alpha}_j} \tag{1-30}$$

Given a Cartan matrix, one can reconstruct the simple roots of the algebra.

Definition 8. *The Dynkin diagram associated with a complex semi-simple Lie algebra of rank* n *is a graph with* n *nodes, each node corresponding to one of the simple roots. If* $i \neq j$ *then the* i*-th and* j*-th node is connected by* $n_{ij} = K_{ij}K_{ji}$ *edges (no sum over repeated indices here).*

If two nodes are connected by more than one edge then an arrow is added in the direction of the shorter root. If two nodes are connected by only one edge, then no arrow is added.

Proposition 7. *If the Lie algebra is simple, then the Dynkin diagram must be connected.*

Proof

Suppose the diagram decomposes into two (or more) disconnected components, then this implies that the simple roots can be split into two non-empty sets R and $R_\perp$ with elements in R orthogonal to those in $R_\perp$ and $R \cap R_\perp = \emptyset$.

Let L_1 denote the Lie subalgebra of L obtained by taking the span of the elements in R and those in the corresponding root spaces $L_{\underline{\alpha}}$ for $\underline{\alpha} \in R$ together with all possible commutators between these elements.

Suppose that $L_1 = L$. Then consider some $\underline{\alpha} \in R_\perp$. The root space $L_{\underline{\alpha}}$ must correspond to a root space $V_{\underline{\beta}}$ for some $\underline{\beta} \in \text{span } R$. Hence

$$\underline{\alpha} = \sum_i k_i \underline{\beta}_i \tag{1-31}$$

for $\underline{\beta}_i \in R$ and $k_i \in \mathbb{N}$. But this is not possible, as $\underline{\alpha}$ and $\underline{\beta}_i$ are simple roots, hence linearly independent. ■

Given such a diagram, one can reconstruct the Cartan matrix. This is because there are only finitely many possible values for $K_{ij}K_{ji}$ and knowing these, together with which roots are shorter than others, is sufficient to fix K_{ij}- e.g. if $K_{ij}K_{ji} = 3$ then one must have $K_{ij} = -1$ and $K_{ji} = -3$ or $K_{ij} = -3$ and $K_{ji} = -1$ and one has $|\underline{\alpha}_i| \neq |\underline{\alpha}_j|$; the direction of the arrow determines which root is longer, the length ratio between the longer and shorter root is $\sqrt{3}$. However, if $K_{ij}K_{ji} = 1$ then $K_{ij} = K_{ji} = -1$ and $|\underline{\alpha}_i| = |\underline{\alpha}_j|$.

From the table of possible values of K_{ij} we observe that two nodes can be linked with at most 3 edges. It is possible to obtain rather severe constraints on possible Dynkin diagrams. In particular, suppose that $\underline{e}_i = \frac{1}{\sqrt{\underline{\alpha}_i.\underline{\alpha}_i}}\underline{\alpha}_i$ are the unit-normalized roots. Then if $i \neq j$, it follows that $\underline{e}_i.\underline{e}_j = -\frac{1}{2}\sqrt{n_{ij}}$

Proposition 8. *A Dynkin diagram has no closed loops, and each vertex meets at most 3 lines.*

Proof Suppose a Dynkin diagram has a closed loop in it, with no subloops. Let the nodes on the rim of the loop be $1, 2, \ldots, k$; as there are no subloops we observe that the only non-vanishing inner products of $\underline{e}_i.\underline{e}_j$ for $i \neq j$ are of the form $\underline{e}_i.\underline{e}_{i+1}$ (with the definition that $\underline{e}_{k+1} = \underline{e}_1$), and $\underline{e}_i.\underline{e}_{i+1} \leq -\frac{1}{2}$.

Set $\underline{\alpha} = \underline{e}_i + \cdots + \underline{e}_k$. Then

$$\underline{\alpha}.\underline{\alpha} = \sum_{i=1}^k \underline{e}_i.\underline{e}_i + 2\sum_{i=1}^k \underline{e}_i.\underline{e}_{i+1} = k + 2\sum_{i=1}^k \underline{e}_i.\underline{e}_{i+1} \leq k + 2(-\frac{1}{2}k) = 0 \tag{1-32}$$

Hence $\underline{\alpha} = 0$. But $\underline{\alpha}$ cannot vanish as the simple roots are linearly independent, which is a contradiction.

Next consider a normalized simple root $\underline{e}$ which is joined to the simple roots $\underline{e}_i$, and $\underline{e}$ is joined to $\underline{e}_i$ by n_i edges. If $i \neq j$ then $\underline{e}_i.\underline{e}_j = 0$ as we have shown that there can be no closed loops. Set $\underline{\beta} = \underline{e} - \sum_i(\underline{e}.\underline{e}_i)\underline{e}_i$. Then $\underline{\beta}.\underline{e}_i = 0$ and hence

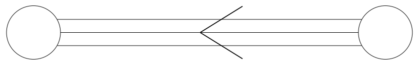
$$\underline{e}.\underline{e} = (\underline{\beta} + \sum_i(\underline{e}.\underline{e}_i)\underline{e}_i).(\underline{\beta} + \sum_j(\underline{e}.\underline{e}_j)\underline{e}_j) = \underline{\beta}.\underline{\beta} + \sum_i(\underline{e}.\underline{e}_i)^2 = \underline{\beta}.\underline{\beta} + \frac{1}{4}\sum_i n_i \tag{1-33}$$

But $\underline{e}.\underline{e} = 1$ hence

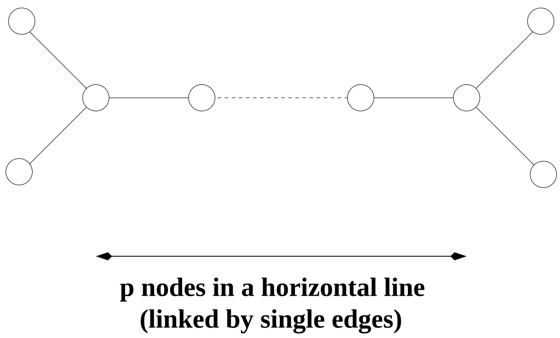
$$1 = \underline{\beta}.\underline{\beta} + \frac{1}{4}\sum_i n_i \tag{1-34}$$

However, as $\underline{\beta}$ cannot vanish (because of the linear independence of the simple roots), $\underline{\beta}.\underline{\beta} > 0$, and hence $\sum_i n_i < 4$. Hence it follows that $\underline{e}$ can be linked to by at most 3 edges. ■.

Note that this means that the only Dynkin diagram which contains a triple edge is that for G_2:



Proposition 9. *It is not possible for a Dynkin diagram to contain the following sub-diagram:*



Proof

Suppose that this sub-diagram is allowed.

Denote the normalized roots corresponding to the horizontal nodes $\underline{e}_1, \dots, \underline{e}_p$ running from left to right. Denote the two normalized roots to the far left by $\underline{\alpha}_1$, $\underline{\alpha}_2$ and the two normalized roots to the far right by $\underline{\beta}_1$ and $\underline{\beta}_2$.

As there are no closed loops we must have $\underline{\alpha}_1.\underline{\alpha}_2 = \underline{\beta}_1.\underline{\beta}_2 = \underline{\alpha}_i.\underline{\beta}_j = 0$; also $\underline{\alpha}_i.\underline{e}_j = 0$ unless $j = 1$, then $\underline{\alpha}_i.\underline{e}_1 = -\frac{1}{2}$ and $\underline{\beta}_i.\underline{e}_j = 0$ unless $j = p$ then $\underline{\beta}_i.\underline{e}_p = -\frac{1}{2}$. The only non-vanishing values for $\underline{e}_i.\underline{e}_j$ for $i \neq j$ are obtained from $\underline{e}_1.\underline{e}_2 = \cdots = \underline{e}_{p-1}.\underline{e}_p = -\frac{1}{2}$.

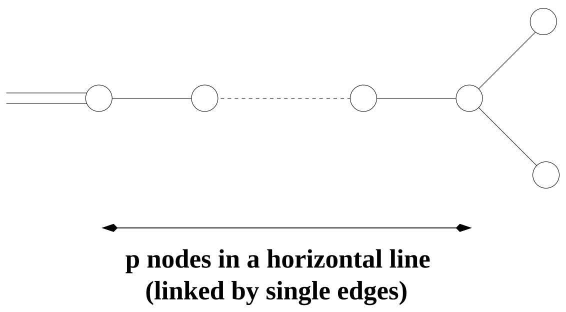
Set $\underline{e} = \underline{e}_1 + \cdots + \underline{e}_p$. Note that $\underline{e}.\underline{e} = 1$.

Also set $\underline{\chi} = \underline{e} + \frac{1}{2}(\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\beta}_1 + \underline{\beta}_2)$. Then $\underline{\chi}.\underline{\alpha}_i = \underline{\chi}.\underline{\beta}_i = 0$, and

$$\begin{aligned} 1 = \underline{e}.\underline{e} &= (\underline{\chi} - \frac{1}{2}(\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\beta}_1 + \underline{\beta}_2)).(\underline{\chi} - \frac{1}{2}(\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\beta}_1 + \underline{\beta}_2)) \\ &= \underline{\chi}.\underline{\chi} + \frac{1}{4}(\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\beta}_1 + \underline{\beta}_2).(\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\beta}_1 + \underline{\beta}_2) \\ &= \underline{\chi}.\underline{\chi} + 1 \end{aligned} \quad \text{(1-35)}$$

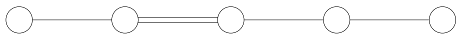
Hence $\underline{\chi}.\underline{\chi} = 0$, which implies $\underline{\chi} = 0$. But $\underline{\chi}$ cannot vanish as the $\underline{e}_i$, $\underline{\alpha}_j$ and $\underline{\beta}_k$ are linearly independent, a contradiction. ■

By essentially analogous reasoning, it is also possible to exclude the following two subdiagrams (double arrows have not been included here)



Exercise: Prove that these two subdiagrams are forbidden.

Proposition 10. *The following subdiagram is forbidden:*



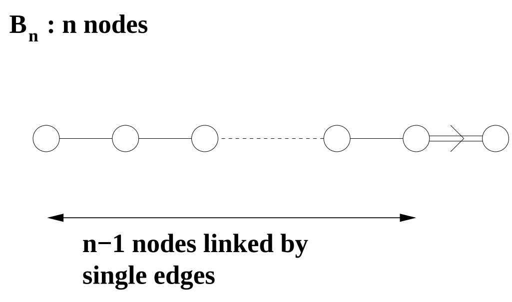
Proof

Label the unit normalized roots $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, $\underline{e}_4$ and $\underline{e}_5$ from left to right.

Set $\underline{e} = \sqrt{2}\underline{e}_1 + 2\sqrt{2}\underline{e}_2 + 3\underline{e}_3 + 2\underline{e}_4 + \underline{e}_5$

Then by direct computation $\underline{e}.\underline{e} = 0$ which implies that $\underline{e} = 0$, in contradiction to the linear independence of $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, $\underline{e}_4$ and $\underline{e}_5$. ■

This classifies all possible diagrams with a double edge; it is not possible for a double edge to appear more than once. They are:

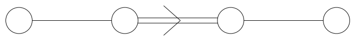


and

C_n : n nodes

and

F_4

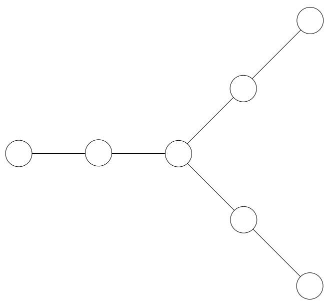


It remains to classify diagrams with only single edges. We have already proven that such a diagram can have at most one branch point. Diagrams with a single branch point can be constrained further:

Proposition 11. *If there is a branch point in a diagram with only single edges then one of the branches has length 1.*

Proof

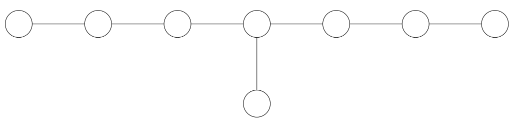
Suppose that this is not the case. Then the diagram contains the sub-diagram:



Suppose that $\underline{\alpha}_1$ is the normalized root at the centre, with $\underline{\alpha}_2$, $\underline{\alpha}_3$, $\underline{\alpha}_4$ the normalized roots adjacent to the centre and $\underline{\alpha}_5$, $\underline{\alpha}_6$, $\underline{\alpha}_7$ the normalized outer roots.

Consider $\underline{\chi} = 3\underline{\alpha}_1 + 2(\underline{\alpha}_2 + \underline{\alpha}_3 + \underline{\alpha}_4) + \underline{\alpha}_5 + \underline{\alpha}_6 + \underline{\alpha}_7$ Then $\underline{\chi} \cdot \underline{\chi} = 0$, a contradiction. $\blacksquare$.

Proposition 12. *The subdiagram*



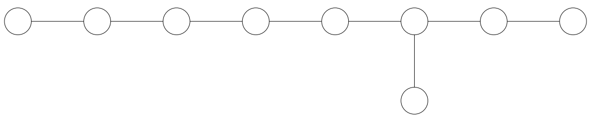
is forbidden.

Proof

Label the horizontal normalized roots from left to right as $\underline{\alpha}_1, \ldots, \underline{\alpha}_7$ with the remaining normalized root $\underline{\alpha}_8$.

Then set $\underline{\chi} = \underline{\alpha}_1 + 2\underline{\alpha}_2 + 3\underline{\alpha}_3 + 4\underline{\alpha}_4 + 3\underline{\alpha}_5 + 2\underline{\alpha}_6 + \underline{\alpha}_7 + 2\underline{\alpha}_8$. Then $\underline{\chi}.\underline{\chi} = 0$, a contradiction. ■

Proposition 13. *The subdiagram*



is forbidden.

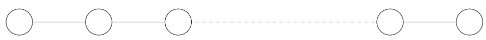
Proof

Label the horizontal normalized roots from left to right as $\underline{\alpha}_1, \ldots, \underline{\alpha}_8$ with the remaining normalized root $\underline{\alpha}_9$.

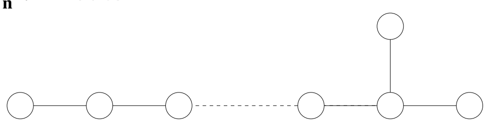
Then set $\underline{\chi} = \underline{\alpha}_1 + 2\underline{\alpha}_2 + 3\underline{\alpha}_3 + 4\underline{\alpha}_4 + 5\underline{\alpha}_5 + 6\underline{\alpha}_6 + 4\underline{\alpha}_7 + 2\underline{\alpha}_8 + 3\underline{\alpha}_9$. Then $\underline{\chi}.\underline{\chi} = 0$, a contradiction. ■

Hence this constrains the possible diagrams involving only single edges to the two classical algebras:

A_n : n nodes

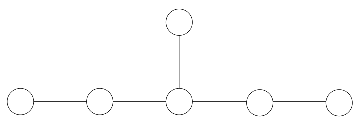


D_n : n nodes

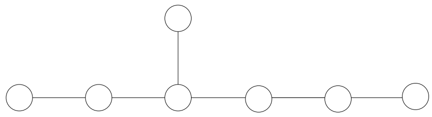


and the three exceptional algebras

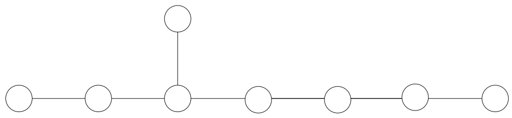
E_6



E_7



E_8



1.6.1 Summary of results so far

We have shown that a semi-simple Lie algebra is completely described by its decomposition into a Cartan subalgebra and root spaces. The simple roots are particularly important, as all roots can be written as integer linear combinations of simple roots, and the structure of the simple roots determines the entire root space. Simple roots of are described entirely by their associated Dynkin diagram (which is connected if the Lie algebra is simple), and we have classified all such Dynkin diagrams.

The remaining steps in classifying the simple Lie algebras are:

i) Show that the structure of the root diagram fixes the commutation relations.

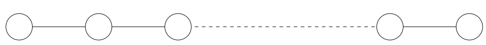
ii) Show that each allowed Dynkin diagram is in fact obtained from one of the classical or exceptional algebras (we have labelled them accordingly, because they do- but we have not proved this!)

In order to show (i) it suffices to obtain a special basis for the Lie algebra in which the structure constants can be read off directly from the properties of the root space. We will not prove this explicitly (the proof is straightforward but rather involved).

To establish (ii) one must compute the root spaces for the classical and exceptional Lie groups. We will not do this for all cases, but rather compute the root space associated with A_n as an example.

1.7 The Lie Algebra A_n

The Lie algebra of A_n has the Dynkin diagram



We will show that this diagram can be obtained from the complexification of $\mathcal{L}(SU(n+1))$, making use of the results of (3.7) in Chapter 3 [corresponding to question 10 of Example Sheet 4.]

The Cartan subalgebra H can be taken to be the traceless diagonal matrices; there are $n(n+1)$ roots associated with the $E_{i,j}$ for $i \neq j$. If the root associated with $E_{i,j}$ is a positive root then the root associated with $E_{j,i}$ is a negative root.

It is useful to construct a basis of H defined by

$$\begin{aligned} e_p &= \frac{1}{\sqrt{2(n+1)p(p+1)}} \sum_{r=1}^{p} r(E_{r,r} - E_{r+1,r+1}) \\ &= \frac{1}{\sqrt{2(n+1)p(p+1)}} \Big(\sum_{r=1}^{p} E_{r,r} - pE_{p+1,p+1} \Big) \end{aligned} \tag{1-36}$$

Then

$$\kappa(e_p, e_q) = \delta_{pq} \tag{1-37}$$

In this basis,

$$[e_p, E_{i,j}] = \alpha_p^{i,j} E_{i,j} \tag{1-38}$$

for $p = 1, \ldots, n$ where

$$\alpha_p^{i,j} = \frac{1}{\sqrt{2(n+1)p(p+1)}} \Big(p(\delta_{j,p+1} - \delta_{i,p+1}) + \sum_{r=1}^{p} (\delta_{i,r} - \delta_{j,r}) \Big) \tag{1-39}$$

denote the root components. One can arrange for an ordering on the space spanned by the roots so that the positive roots are $\alpha^{i,j}$ for $i < j$, to see this note that if $\lambda \in \mathbb{R}^n$ and $i < j$ then

$$\lambda.\alpha^{i,j} = \frac{1}{\sqrt{2(n+1)}} \Big(\sum_{r=i}^{n} \frac{\lambda_r}{\sqrt{r(r+1)}} - \sqrt{1 - \frac{1}{i}}\lambda_{i-1} - \sum_{r=j}^{n} \frac{\lambda_r}{\sqrt{r(r+1)}} + \sqrt{1 - \frac{1}{j}}\lambda_{j-1} \Big) \tag{1-40}$$

(where if $i = 1$ the term $\sqrt{1 - \frac{1}{i}}\lambda_{i-1}$ is taken to vanish). So setting $\lambda_p = \sqrt{\frac{p+1}{p}}$ we see that

$$\lambda.\alpha^{i,j} \geq \frac{1}{\sqrt{2(n+1)}} \sum_{r=i}^{j} \frac{1}{r} > 0 \tag{1-41}$$

and so if $i > j$ then $\lambda.\alpha^{i,j} < 0$. The positive roots $\alpha^{i,j}$ for $i < j$ can be written as positive integer linear sums of $\beta_i = \alpha^{i,i+1}$ for $i = 1, \ldots, n$.

The β_i are the simple roots, which are linearly independent, and have components (for $p = 1, \ldots, n$)

$$(\beta_i)_p = \frac{1}{\sqrt{2(n+1)p(p+1)}}(-p\delta_{p,i-1} + (p+1)\delta_{p,i}) \quad i = 1, \ldots, n \tag{1-42}$$

It is then straightforward to verify that

$$\beta_j.\beta_j = \frac{1}{n+1} \quad j = 1, \ldots, n \tag{1-43}$$

and

$$\beta_i.\beta_j = -\frac{\delta_{i,j-1}}{2(n+1)} \quad 1 \leq i < j \leq n+1 \tag{1-44}$$

This produces the Dynkin diagram for A_n.

1.8 Real Forms and the Chevalley Basis

Definition 9. *Let L be a complex Lie algebra. A real basis for L consists of a basis for L for which all the structure constants are real.*

Definition 10. *Let L be a complex Lie algebra. Suppose that $\{T_a\}$ is a real basis for L. Let L' be the vector space obtained from the span of the T_a over $\mathbb{R}$. Then as the structure constants with respect to the basis T_a are real, it follows that L' is a Lie algebra. L' is called the real form of L. The Lie algebra L is obtained by complexifying L'.*

Definition 11. *Let L be a complex Lie algebra with real basis $\{T_a\}$. Let L' be the real Lie algebra obtained from the span of the T_a over $\mathbb{R}$. Then L' is called a compact real form if the Killing form of L' is negative definite.*

Theorem 2. *Let L be a semisimple complex Lie algebra. Then there exists a basis of L consisting of $\{h_{\underline{\alpha}_i} : i = 1, \ldots, n\}$ and $\{J_{\underline{\alpha}}\}$ where $\underline{\alpha}_i$ are the simple roots and $\underline{\alpha}$ are roots satisfying*

$$\begin{aligned}
[h_{\underline{\alpha}_i}, h_{\underline{\alpha}_j}] &= 0 \\
[h_{\underline{\alpha}_i}, J_{\underline{\alpha}}] &= (\frac{2\underline{\alpha}_i.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}})J_{\underline{\alpha}} \\
[J_{\underline{\alpha}}, J_{-\underline{\alpha}}] &= h_{\underline{\alpha}}
\end{aligned} \tag{1-45}$$

where $h_{\underline{\alpha}}$ is an integer linear combination of the $h_{\underline{\alpha}_i}$; and if $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\alpha} + \underline{\beta}$ are roots

$$[J_{\underline{\alpha}}, J_{\underline{\beta}}] = \pm(1-p)J_{\underline{\alpha}+\underline{\beta}} \tag{1-46}$$

where p is the smallest integer such that $\underline{\beta} + p\underline{\alpha}$ is a root.

This basis is called the Chevalley basis.

Proof

We have already established much of this. In particular, as the $\underline{\alpha}_i$ are linearly independent on $\mathbb{R}^n$ it follows that the $h_{\underline{\alpha}_i}$ are linearly independent on H and therefore form a basis for H. It remains to establish (1-46). In general, from our previous reasoning, if $\underline{\alpha}$, $\underline{\beta}$ are roots with $\underline{\alpha} \pm \underline{\beta} \neq 0$, then by constructing the $\underline{\alpha}$ string through $\underline{\beta}$ we find

$$[J_{\underline{\alpha}}, J_{\underline{\beta}}] = N_{\underline{\alpha},\underline{\beta}} J_{\underline{\alpha}+\underline{\beta}} \tag{1-47}$$

for some $N_{\underline{\alpha},\underline{\beta}} \in \mathbb{C}$ with $N_{\underline{\alpha},\underline{\beta}} \neq 0$ if $\underline{\alpha} + \underline{\beta}$ is a root, and $N_{\underline{\alpha},\underline{\beta}} = 0$ otherwise. Next note that the commutation relations (1-45) are left invariant under the re-scaling $J_{\underline{\alpha}} \rightarrow c_{\underline{\alpha}} J_{\underline{\alpha}}$ provided that $c_{\underline{\alpha}} c_{-\underline{\alpha}} = 1$ for all roots $\underline{\alpha}$. This re-scales $N_{\underline{\alpha},\underline{\beta}}$ as $N_{\underline{\alpha},\underline{\beta}} \rightarrow c_{-\underline{\alpha}} c_{-\underline{\beta}} N_{\underline{\alpha},\underline{\beta}}$. We will not present the remainder of the argument here; it involves establishing some additional constraints on the $N_{\underline{\alpha},\underline{\beta}}$, such as $N_{\underline{\alpha},\underline{\beta}} = N_{-\underline{\alpha},-\underline{\beta}}$, which are sufficient, together with the rescaling freedom, to establish (1-46) (the details can be found in Samelson). ■

Proposition 14. *If L is a semi-simple complex Lie algebra then L has a compact real form.*

Proof

The compact Lie form can be constructed from the Chevalley basis by taking a new basis given by $ih_{\underline{\alpha}_i}$, $i(J_{\underline{\alpha}} + J_{-\underline{\alpha}})$ and $J_{\underline{\alpha}} - J_{-\underline{\alpha}}$.

Then

$$\begin{aligned}
[ih_{\underline{\alpha}_i}, ih_{\underline{\alpha}_j}] &= 0 \\
[ih_{\underline{\alpha}_i}, i(J_{\underline{\alpha}} + J_{-\underline{\alpha}})] &= -(\frac{2\underline{\alpha}_i.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}})(J_{\underline{\alpha}} - J_{-\underline{\alpha}}) \\
[ih_{\underline{\alpha}_i}, J_{\underline{\alpha}} - J_{-\underline{\alpha}}] &= (\frac{2\underline{\alpha}_i.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}})i(J_{\underline{\alpha}} + J_{-\underline{\alpha}})
\end{aligned} \tag{1-48}$$

and if $\underline{\alpha} \pm \underline{\beta} \neq 0$ then

$$\begin{aligned}
[i(J_{\underline{\alpha}} + J_{-\underline{\alpha}}), J_{\underline{\beta}} - J_{-\underline{\beta}}] &= iN_{\underline{\alpha},\underline{\beta}} J_{\underline{\alpha}+\underline{\beta}} - iN_{-\underline{\alpha},-\underline{\beta}} J_{-\underline{\alpha}-\underline{\beta}} + iN-\underline{\alpha}, \underline{\beta} J_{-\underline{\alpha}+\underline{\beta}} - iN_{\underline{\alpha},-\underline{\beta}} J_{\underline{\alpha}-\underline{\beta}} \\
&= iN_{\underline{\alpha},\underline{\beta}}(J_{\underline{\alpha}+\underline{\beta}} + J_{-(\underline{\alpha}+\underline{\beta})}) + IN_{-\underline{\alpha},\underline{\beta}}(J_{-\underline{\alpha}+\underline{\beta}} + J_{\underline{\alpha}-\underline{\beta}})
\end{aligned} \tag{1-49}$$

and

$$[i(J_{\underline{\alpha}} + J_{-\underline{\alpha}}), J_{\underline{\alpha}} - J_{-\underline{\alpha}}] = -2i[J_{\underline{\alpha}}, J_{-\underline{\alpha}}] = -2ih_{\underline{\alpha}} \tag{1-50}$$

where $h_{\underline{\alpha}}$ is an integer linear combination of the $h_{\underline{\alpha}_i}$.

This is sufficient to show that there is a basis for which the structure constants are real.

Next we must consider the Killing form restricted to the linear span of this basis over $\mathbb{R}$.

We have already proven that κ is positive definite when restricted to the real span of the $h_{\underline{\alpha}_i}$. So κ is negative definite when restricted to the real span of the $ih_{\underline{\alpha}_i}$. In addition,

$$\kappa(h, J_{\underline{\alpha}}) = 0 \tag{1-51}$$

for all $h \in H$ and roots $\underline{\alpha}$. Also, if $\underline{\alpha} \pm \underline{\beta} \neq 0$ then

$$\kappa(i(J_{\underline{\alpha}}+J_{-\underline{\alpha}}), J_{\underline{\beta}}-J_{-\underline{\beta}}) = \kappa(i(J_{\underline{\alpha}}+J_{-\underline{\alpha}}), i(J_{\underline{\beta}}+J_{-\underline{\beta}})) = \kappa(J_{\underline{\alpha}}-J_{-\underline{\alpha}}, J_{\underline{\beta}}-J_{-\underline{\beta}}) = 0 \tag{1-52}$$

Also, recall that

$$\kappa(J_{\pm\underline{\alpha}}, J_{\pm\underline{\alpha}}) = 0, \qquad \kappa(J_{\underline{\alpha}}, J_{-\underline{\alpha}}) = \frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} \tag{1-53}$$

Hence

$$\kappa((i(J_{\underline{\alpha}} + J_{-\underline{\alpha}}), J_{\underline{\alpha}} - J_{-\underline{\alpha}})) = 0 \tag{1-54}$$

and

$$\kappa(i(J_{\underline{\alpha}} + J_{-\underline{\alpha}}), i(J_{\underline{\alpha}} + J_{-\underline{\alpha}})) = \kappa(J_{\underline{\alpha}} - J_{-\underline{\alpha}}, J_{\underline{\alpha}} - J_{-\underline{\alpha}}) = -\frac{2}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} \tag{1-55}$$

So, in this basis, κ is negative definite. ■.

1.9 Representations and Weights

We shall use the techniques which we have developed to classify complex semisimple Lie algebras in order to investigate representations of complex semisimple Lie algebras acting on a vector space V.

Let d denote a representation of a semi-simple complex Lie algebra L of rank n, acting on V (where V is a vector space over $\mathbb{C}$). Let H denote the Cartan subalgebra of L. If $h_{\underline{\alpha}} \in H$, $J_{\underline{\alpha}} \in L_{\underline{\alpha}}$ are as in the previous section, then $d(h_{\underline{\alpha}})$ and $d(J_{\pm\underline{\alpha}})$ generate a $\mathcal{L}(SU(2))$ algebra acting on V. It follows that $d(h_{\underline{\alpha}})$ is diagonalizable over V. As all elements of H commute with each other, and the $h_{\underline{\alpha}}$ span H, it follows that $d(h_i)$ can be simultaneously diagonalized for all roots $h_i \in H$.

Definition 12. *$\underline{w}$ is a weight of the representation d if there exists $v \in V$, $v \neq 0$ such that $d(h_i)v = \underline{w}_i v$ where h_i is a basis of H.*

The weights $\underline{w}$ are the simultaneous eigenvectors of all the $d(h_{\underline{\alpha}})$. Given a weight $\underline{w}$, define

$$V_{\underline{w}} = \{v \in V : d(h_i)v = \underline{w}_i v\} \tag{1-56}$$

As the $d(h_{\underline{\alpha}})$ can all be simultaneously diagonalized it follows that

$$V = \bigoplus_{\text{weights } \underline{w}} V_{\underline{w}} \tag{1-57}$$

Proposition 15. *Let $\underline{w}$ be a weight of the representation d, and suppose that $\underline{\alpha}$ is a root. Suppose $v \in V_{\underline{w}}$. If $d(J_{\underline{\alpha}})v \neq 0$, then $d(J_{\underline{\alpha}})v$ is a simultaneous eigenstate of $d(h_i)$ with weight $\underline{w} + \underline{\alpha}$.*

Proof

This is established by noting that

$$\begin{aligned} d(h_i)d(J_{\underline{\alpha}})v &= [d(h_i), d(J_{\underline{\alpha}})]v + d(J_{\underline{\alpha}})d(h_i)v \\ &= d([h_i, J_{\underline{\alpha}}])v + d(J_{\underline{\alpha}})\underline{w}_i v \\ &= \underline{\alpha}_i d(J_{\underline{\alpha}})v + \underline{w}_i d(J_{\underline{\alpha}})v \\ &= (\underline{w} + \underline{\alpha})_i d(J_{\underline{\alpha}})v \end{aligned} \tag{1-58}$$

Hence if $d(J_{\underline{\alpha}})v \neq 0$ then $\underline{w} + \underline{\alpha}$ is also a weight. ■

Proposition 16. *Let $\underline{w}$ be a weight of the representation d, and suppose that $\underline{\alpha}$ is a root. Then the vectors $\underline{w} + n\underline{\alpha}$ are weights for all $p \leq n \leq q$ for some $p, q \in \mathbb{Z}$, $p \leq 0 \leq q$.*

In addition $\frac{2\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = -(p+q)$, and $\underline{w} - \frac{2\underline{\alpha}.\underline{w}}{\underline{\alpha}.\underline{\alpha}}\underline{\alpha}$ is a weight.

Proof

Let $v \in V_{\underline{w}}$. Consider the vector space W obtained from the span of v and $(J_{\pm\underline{\alpha}})^m v$ for $m \in \mathbb{N}$. This is invariant under the action of $d(h_{\underline{\alpha}})$ and $d(J_{\pm\underline{\alpha}})$. If $(J_{\pm\underline{\alpha}})^m v \neq 0$ then $(J_{\pm\underline{\alpha}})^m v$ is an eigenstate of $d(h_i)$ with weight $\underline{w} \pm m\underline{\alpha}$. Hence $d(h_{\underline{\alpha}})$ and $d(J_{\pm\underline{\alpha}})$ generate an irreducible representation of $\mathcal{L}(SU(2))$ acting on W. So there exist integers p, q with $p \leq 0 \leq q$ such that $\underline{w} + n\underline{\alpha}$ is a weight if $n \in bZ$, $p \leq n \leq q$.

Suppose that $v' \in V_{\underline{w}+n\underline{\alpha}}$ for such n. We continue to work in a basis $\{h_i\}$ of H for which $\kappa(h_i, h_j) = \delta_{ij}$, and $\alpha^i = y^i_{\underline{\alpha}} \in \mathbb{R}$.

Then

$$\begin{aligned} d(h_{\underline{\alpha}})v' &= d(\frac{1}{\kappa(y_{\underline{\alpha}}, y_{\underline{\alpha}})} y_{\underline{\alpha}})v' \\ &= \frac{\alpha^i}{\underline{\alpha}.\underline{\alpha}} d(h_i)v' \\ &= \frac{\underline{\alpha}.(\underline{w} + n\underline{\alpha})}{\underline{\alpha}.\underline{\alpha}} v' \end{aligned} \tag{1-59}$$

The largest and smallest of the possible eigenvalues is $\pm r$ where $2r \in \mathbb{N}$.

Hence

$$q + \frac{\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = r, \qquad p + \frac{\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = -r \tag{1-60}$$

and therefore

$$\frac{2\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} = -(p+q) \tag{1-61}$$

In particular, this implies that $p \leq -\frac{2\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}} \leq q$. Hence $\underline{w} - \frac{2\underline{\alpha}.\underline{w}}{\underline{\alpha}.\underline{\alpha}}\underline{\alpha}$ is a weight. ■

Note that this implies that in the basis for H in which the Killing form and root components are real, the weight components must also be real, so one can regard the weights as vectors in $\mathbb{R}^n$.

Definition 13. *A weight $\underline{w}$ is a highest weight if $\underline{w}+\underline{\alpha}$ is not a weight for any positive root $\underline{\alpha}$.*

Proposition 17. *If d is a representation of a complex semisimple Lie algebra acting on the complex finite-dimensional vector space V, then there exists a highest weight.*

Proof

We have shown that there exists at least one weight $\underline{\chi}$, with some $v \in V$, $v \neq 0$ such that $d(h_i)v = \underline{\chi}_i v$. Consider constructing all possible weights of the form $\underline{\chi}+\underline{\alpha}$ for positive roots $\underline{\alpha}$ by acting on v by $d(J_{\underline{\alpha}})$. Then repeat this process. Eventually this process must stop, otherwise, if it continued indefinitely, one could construct arbitrarily many linearly independent vectors in V, in contradiction with the finite-dimensionality of V. Hence there must exist a weight $\underline{w}$ such that $\underline{w}+\underline{\alpha}$ is not a weight for all positive roots $\underline{\alpha}$. ■

Proposition 18. *If $\underline{w}$ is a maximal weight and $\underline{\alpha}$ is a simple root then $\underline{w}.\underline{\alpha} \geq 0$.*

Proof

We have shown that $\underline{w} - \frac{2\underline{w}.\underline{\alpha}}{\underline{\alpha}.\underline{\alpha}}\underline{\alpha}$ is a weight. If $\underline{\alpha}.\underline{w} > 0$ then this implies that the $\underline{\alpha}$ string of weights passing through $\underline{w}$ contains $\underline{w}+\underline{\alpha}$, i.e. $\underline{w}+\underline{\alpha}$ is a weight. This is a contradiction, as $\underline{w}$ is a highest weight. ■

Proposition 19. *Suppose that $\underline{w}$ is a highest weight, and let $v \in V_{\underline{w}}$, $v \neq 0$. Define V' to be the span of v and all possible products of $\prod d(J_{-\underline{\alpha}})v$ for simple roots $\underline{\alpha}$. Then V' is an invariant subspace of V.*

Proof

Note that all vectors $\prod d(J_{-\underline{\alpha}})v$ either vanish or are eigenstates of $d(h_i)$ and hence of $d(h_{\underline{\beta}})$ for all roots $\underline{\beta}$.

Suppose that $\underline{\beta}$ is a positive root. Then one can write $\underline{\beta} = \underline{\beta}_1 + \cdots + \underline{\beta}_k$ for simple roots $\underline{\beta}_1$, and $\underline{\beta}_1 + \cdots + \underline{\beta}_k$ is a root for $j = 1, \ldots, k$. It follows that

$$J_{\underline{\beta}} = \lambda[J_{\underline{\beta}_k}, [J_{\underline{\beta}_{k-1}} \ldots, J_{\underline{\beta}_1}] \ldots] \tag{1-62}$$

and

$$J_{-\underline{\beta}} = \mu[J_{-\underline{\beta}_k}, [J_{-\underline{\beta}_{k-1}} \ldots, J_{-\underline{\beta}_1}] \ldots] \tag{1-63}$$

for some constants λ, μ. Hence

$$d(J_{\underline{\beta}}) = \lambda[d(J_{\underline{\beta}_k}), [d(J_{\underline{\beta}_{k-1}}) \ldots, d(J_{\underline{\beta}_1})] \ldots] \tag{1-64}$$

and

$$d(J_{-\underline{\beta}}) = \mu[d(J_{-\underline{\beta}_k}), [d(J_{-\underline{\beta}_{k-1}}) \ldots, d(J_{-\underline{\beta}_1})] \ldots] \tag{1-65}$$

As V' is invariant under $d(J_{-\underline{\beta}_i})$ by construction, it follows that V' is invariant under $d(J_{-\underline{\beta}})$. Also, to establish that $d(J_{\underline{\beta}})V' \subset V'$ it suffices to show that $d(J_{\underline{\beta}})V' \subset V'$ for simple roots $\underline{\beta}$.

Suppose then that $\underline{\beta}$ is a simple root. V' is spanned by $\prod_{i=1}^{k} d(J_{-\underline{\alpha}_i})v$ where $\underline{\alpha}_i$ are simple. First note that $d(J_{\underline{\beta}})v = 0$ as $\underline{w} + \underline{\beta}$ is not a weight.

It therefore suffices to establish that $d(J_{\underline{\beta}}) \prod_{i=1}^{k} d(J_{-\underline{\alpha}_i})v \in V'$ for simple roots $\underline{\alpha}_i$.

We do this by induction on k. This is true for $k = 1$, because

$$d(J_{\underline{\beta}})d(J_{-\underline{\alpha}_1})v = [d(J_{\underline{\beta}}), d(J_{-\underline{\alpha}_1})]v = d([J_{\underline{\beta}}, J_{-\underline{\alpha}_1}])v \tag{1-66}$$

This vanishes unless $\underline{\beta} = \underline{\alpha}_1$, in which case $[J_{\underline{\beta}}, J_{-\underline{\alpha}_1}] \propto d(h_{\underline{\beta}})$ and $d(h_{\underline{\beta}})v = \mu v$ for some constant μ.

Suppose it is true for $k = \ell$. Consider

$$d(J_{\underline{\beta}}) \prod_{i=1}^{\ell+1} d(J_{-\underline{\alpha}_i})v = d(J_{\underline{\beta}})d(J_{-\underline{\alpha}_1})v' \tag{1-67}$$

where $v' = \prod_{i=2}^{\ell+1} d(J_{-\underline{\alpha}_i})v \in V'$ is an eigenstate of $d(h_i)$.

However,

$$\begin{aligned} d(J_{\underline{\beta}})d(J_{-\underline{\alpha}_1})v' &= [d(J_{\underline{\beta}}), d(J_{-\underline{\alpha}_1})]v' + d(J_{-\underline{\alpha}_1})d(J_{\underline{\beta}})v' \\ &= d([J_{\underline{\beta}}, J_{-\underline{\alpha}_1}])v' + d(J_{-\underline{\alpha}_1})d(J_{\underline{\beta}})v' \end{aligned} \tag{1-68}$$

Now, from the induction, $d(J_{\underline{\beta}})v' \in V'$ and hence $d(J_{-\underline{\alpha}_1})d(J_{\underline{\beta}})v' \in V'$. Also, $d([J_{\underline{\beta}}, J_{-\underline{\alpha}_1}])v'$ vanishes unless $\underline{\beta} = \underline{\alpha}_1$, in which case $[J_{\underline{\beta}}, J_{-\underline{\alpha}_1}] \propto d(h_{\underline{\beta}})$ and $d(h_{\underline{\beta}})v' = \nu v'$ for some constant ν. If follows that $d(J_{\underline{\beta}})d(J_{-\underline{\alpha}_1})v' \in V'$. This establishes the induction. ■

From this construction of a d-invariant subspace from a state of highest weight, one immediately finds the

Corollory 4. *Suppose that d is an irreducible representation acting on V. Then there exists a unique highest weight with multiplicity 1.*

Proof

Suppose that $\underline{w}$ and $\underline{w}'$ are two highest weights with $\underline{w} \neq \underline{w}'$. Let v and $v' \in V$ be eigenstates corresponding to these two weights respectively.

We have shown that by acting on v with all lowering operators $d(J_{-\underline{\alpha}})$ for positive roots $\underline{\alpha}$, one obtains an invariant subspace of V. As d is irreducible, this invariant subspace must be V. But $\underline{w}'$ is a weight of d, so $\underline{w}' = \underline{w} - \underline{\beta}$ where $\underline{\beta}$ is a positive integer linear combination of positive roots.

However, by exactly the same reasoning, one must also have $\underline{w} = \underline{w}' - \underline{\beta}'$ where $\underline{\beta}'$ is also a positive integer linear combination of positive roots. Hence $\underline{\beta} + \underline{\beta}' = 0$, a contradiction, as $\underline{\beta}$ and $\underline{\beta}'$ are both positive vectors.

Hence it follows that there is only one highest weight $\underline{w}$. If v is a corresponding eigenstate associated with this weight, then one generates a spanning set for the whole of V by by acting on v with all lowering operators $d(J_{-\underline{\alpha}})$ for positive roots $\underline{\alpha}$. It follows that one can construct a basis for V consisting of v with weight $\underline{w}$ and other vectors with weights $\underline{w} - \underline{\beta}$ where $\underline{\beta}$ is a sum of positive roots. Hence the weight space corresponding to $\underline{w}$ is spanned by v, and is one-dimensional. ■

We have shown that all weights $\underline{w}$ are constrained by $\frac{2\underline{\alpha}.\underline{w}}{\underline{\alpha}.\underline{\alpha}} \in \mathbb{Z}$ for all roots $\underline{\alpha}$. This forces the weights to lie in a lattice in $\mathbb{R}^n$ called the weight lattice. The roots also lie in a lattice defined by integer linear combinations of the simple roots. Hence, the difference of any two weights of an irreducible representation corresponds to a sum of roots, which therefore lies in the root lattice.

Definition 14. *The fundamental weights $\underline{w}_i$ satisfy $\frac{2\underline{\alpha}_j.\underline{w}_i}{\underline{\alpha}_j.\underline{\alpha}_j} = \delta_{ij}$ where $\underline{\alpha}_j$ is the j-th simple root.*

The fundamental weights are linearly independent vectors in $\mathbb{R}^n$, so if $\underline{w}$ is a weight one can write

$$\underline{w} = \sum n_i \underline{w}_i \tag{1-69}$$

for constants n_i. If $\underline{\alpha}_j$ is the j-th simple root then

$$n_i = \frac{2\underline{w}.\underline{\alpha}_i}{\underline{\alpha}_i.\underline{\alpha}_i} \in \mathbb{Z} \tag{1-70}$$

Hence the fundamental weights form a basis for the weight lattice. All weights can be written as certain integer linear combinations of the fundamental weights.

Definition 15. *The dominant weights I^d consist of those elements $\underline{w}$ of the weight lattice such that $\underline{w}.\underline{\alpha} \geq 0$ for all positive roots $\underline{\alpha}$.*

We have shown that if $\underline{w}$ is a highest weight then $\underline{w} \in I^d$. The converse is also true. We shall state, but not prove the following two important theorems

Theorem 3. *Every irreducible representation of a complex semisimple Lie algebra L on a finite-dimensional complex vector space has a unique highest weight $\underline{w} \in I^d$. Conversely, given any $\underline{w} \in I^d$ there exists an irreducible representation of L with highest weight $\underline{w}$. If two irreducible representations have the same highest weight they are equivalent.*

Theorem 4. *Let d be a representation of a complex semisimple Lie algebra L acting on a finite-dimensional complex vector space V. Then V can be decomposed as $V = V_1 \oplus \cdots \oplus V_k$ where the V_i are invariant subspaces of V with respect to d, and d restricted to V_i is irreducible.*

In this way, the representations of L acting on V are classified entirely (up to equivalence) by their highest weights. In particular, given a knowledge of the simple roots $\underline{\alpha}_i$ of L, one can construct the fundamental weights $\underline{w}_i$. The dominant weights consist of $I^d = \{\sum_{i=1}^{n} m_i \underline{w}_i : m_i \in \mathbb{N}\}$.

So, the highest weight can be written $\underline{w} = \sum n_i \underline{w}_i$ for $n_i \in \mathbb{N}$. The highest weight (and therefore the entire representation) is therefore determined by the non-negative integers n_i. This information can be appended to the Dynkin diagram to classify the representation: i.e. one writes the non-negative integer n_i next to the node corresponding to the i-th simple root. This then fixes the highest weight and therefore the representation.

1.9.1 The weights of $\mathcal{L}(SU(3))$

As an example, we examine the weight lattice of $SU(3)$. From our previous examination of the structure of A_2, we have computed the simple roots

$$\underline{\alpha}_1 = (\frac{1}{\sqrt{3}}, 0), \qquad \underline{\alpha}_2 = (-\frac{1}{2\sqrt{3}}, \frac{1}{2}) \tag{1-71}$$

The corresponding fundamental weights are

$$\underline{w}_1 = (\frac{1}{2\sqrt{3}}, \frac{1}{6}), \qquad \underline{w}_2 = (0, \frac{1}{3}) \tag{1-72}$$

Observe that $\underline{\alpha}_1 = 2\underline{w}_1 - \underline{w}_2$, $\underline{\alpha}_2 = -\underline{w}_1 + 2\underline{w}_2$.

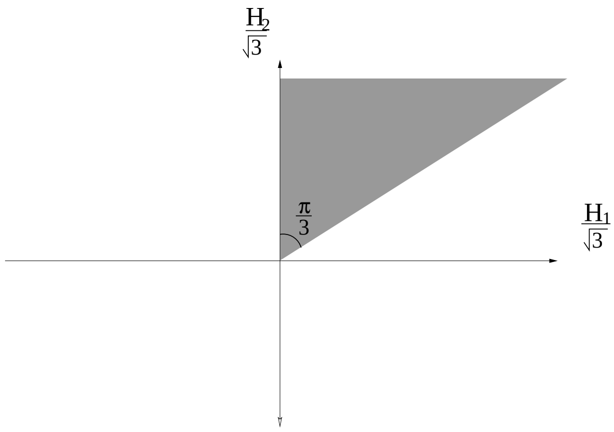
Suppose that $\underline{w}$ is the highest weight $\underline{w} = n_1 \underline{w}_1 + n_2 \underline{w}_2$ for some n_1, $n_2 \in \mathbb{N}$. Then if $\underline{\chi}$ is *any* weight of the same representation, $\underline{\chi} = m_1 \underline{w}_1 + m_2 \underline{w}_2$ for some m_1, $m_2 \in \mathbb{N}$ the difference $\underline{w} - \underline{\chi}$ lies in the root lattice, i.e.

$$\begin{aligned} (n_1 - m_1)\underline{w}_1 + (n_2 - m_2)\underline{w}_2 &= k_1(2\underline{w}_1 - \underline{w}_2) + k_2(-\underline{w}_1 + 2\underline{w}_2) \\ &= (2k_1 - k_2)\underline{w}_1 + (2k_2 - k_1)\underline{w}_2 \end{aligned} \tag{1-73}$$

for some $k_1, k_2 \in \mathbb{N}$. Hence $(n_1 - m_1) - (n_2 - m_2) = 3(k_1 - k_2) \equiv 0 \bmod 3$.

We therefore find $m_1 - m_2 \equiv n_1 - n_2 \bmod 3$. This result is called triality- all weights associated with a representation of $\mathcal{L}(SU(3))$ have the same triality.

The highest weights computed here are of the form $\underline{w} = n_1(\frac{1}{2\sqrt{3}}, \frac{1}{6}) + n_2(0, \frac{1}{3})$ for $n_1, n_2 \in \mathbb{N}$, which when plotted in $\mathbb{R}^2$ occupy a sector subtending an angle $\frac{\pi}{3}$ at the origin:



Note: The careful reader will observe that these roots and weights appear to be different from those computed previously, by a factor of $\frac{1}{\sqrt{3}}$. This is because the components of the roots and weights are defined with respect to some basis of the Cartan subalgebra. We have worked with an orthonormal basis e_1, e_2 satisfying $\kappa(e_i, e_j) = \delta_{ij}$. It is straightforward to see that this basis is related to the basis h_1, h_2 of the Cartan subalgebra which we used before by a rescaling $e_i = \frac{1}{\sqrt{3}} h_i$, which accounts for the scale difference.

Finally, we can write down the Dynkin diagrams corresponding to representations of $SU(3)$: the generic diagram is

which corresponds to a representation with highest weight $n\underline{w}_1 + m\underline{w}_2$.

The complex conjugate representation to this has Dynkin diagram

The adjoint representation is real (i.e. it is its own complex conjugate), and corresponds to

The representations with triangular weight diagrams are

and


0
n