Examples Sheet 3

1. Schur's Lemma is "Let D be an irreducible representation of a (Lie) group G acting on a complex vector space V. Let A be an operator acting on V which commutes with the action of G, that is, $AD(g) = D(g)A$ for all $g \in G$. Then $A = \lambda I_V$, where λ is a constant and I_V is the unit operator."

   Prove this by showing that any eigenspace of A is an invariant subspace of V, and that there is therefore precisely one eigenspace of A which is the whole of V, and that this gives the desired result.

2. The following multiplication rule will be useful in this question (cf. Sheet 1, 3c):

$$(aI + \mathbf{b}\cdot\boldsymbol{\sigma})(a'I + \mathbf{b}'\cdot\boldsymbol{\sigma}) = (aa' + \mathbf{b}\cdot\mathbf{b}')I + (a\mathbf{b}' + a'\mathbf{b} + i\mathbf{b}\times\mathbf{b}')\cdot\boldsymbol{\sigma}\,.$$

   (a) Show how $B(\psi, \mathbf{n}) \in SL(2, \mathbb{C})$, where

$$B(\psi, \mathbf{n}) = I\cosh\frac{\psi}{2} + \boldsymbol{\sigma}\cdot\mathbf{n}\,\sinh\frac{\psi}{2}\,, \qquad \mathbf{n}\cdot\mathbf{n} = 1\,,$$

   corresponds to a Lorentz boost with velocity $\mathbf{v} = \tanh\psi\,\mathbf{n}$.

   (b) Show that

$$\big(I + \frac{1}{2}\boldsymbol{\sigma}\cdot\delta\mathbf{v}\big)B(\psi, \mathbf{n}) = B(\psi', \mathbf{n}')R,$$

   where, to first order in $\delta\mathbf{v}$,

$$\psi' = \psi + \delta\mathbf{v}\cdot\mathbf{n}, \quad \mathbf{n}' = \mathbf{n} + \coth\psi[\delta\mathbf{v} - \mathbf{n}(\mathbf{n}\cdot\delta\mathbf{v})],$$

   and R is an infinitesimal rotation given by

$$R = I + \frac{i}{2}\tanh\frac{\psi}{2}(\delta\mathbf{v}\times\mathbf{n})\cdot\boldsymbol{\sigma} = I + \frac{i}{2}\frac{\gamma}{\gamma+1}(\delta\mathbf{v}\times\mathbf{v})\cdot\boldsymbol{\sigma}, \qquad \gamma = (1-\mathbf{v}^2)^{-\frac{1}{2}}.$$

   (c) Show that we must have $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\delta\mathbf{v})$.

   (d) By considering boosts by velocities $\mathbf{v}, \mathbf{w}$ followed by boosts by $-\mathbf{v}$, $-\mathbf{w}$, find a physical interpetation of this question.

3. A field $\phi(x)$ transforms under the action of a Poincaré transformation (Λ, a) such that $U[\Lambda, a]\phi(x)U[\Lambda, a]^{-1} = \phi(\Lambda x + a)$. For an infinitesimal transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and correspondingly (in physics conventions) $U[\Lambda, a] = 1 - i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} - ia^\mu P_\mu$.

   (a) Show that

$$[M_{\mu\nu}, \phi(x)] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi(x)\,, \qquad [P_\mu, \phi(x)] = i\partial_\mu\phi(x)\,.$$

(b) Verify that $M_{\mu\nu} \to i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ and $P_\mu \to -i\partial_\mu$ satisfy the algebra for $[M_{\mu\nu}, M_{\sigma\rho}]$ and $[M_{\mu\nu}, P_\sigma]$ expected for the Poincaré group.

4. Consider the little group with standard momentum $k^\mu = (\ell, 0, 0, \ell)$, for some fixed $\ell > 0$, that is, the subgroup of proper, orthochronous Lorentz transformations which leaves k^μ invariant.

   (a) Show how the generators of the little group are related to the generators of $SO(1,3)^\uparrow$. [Hint: It will be convenient to define $E_1 := K_1 - J_2$ and $E_2 := K_2 + J_1$.] Find the structure constants of the corresponding Lie algebra and determine whether it is semisimple. [Note: this group is $ISO(2)$, the isometry group of the plane, or the 2-dimensional Euclidean group.]

   (b) Prove that, for appropriately normalized generators,
$$e^{\theta J_3}(a_1 E_1 + a_2 E_2)e^{-\theta J_3} = \alpha_1(\theta)E_1 + \alpha_2(\theta)E_2\,,$$
   where $\theta, a_1, a_2 \in \mathbb{R}$ and
$$\begin{pmatrix} \alpha_1(\theta) \\ \alpha_2(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

   (c) Defining a unitary operator $O[\theta, a_1, a_2] = e^{a_1 E_1 + a_2 E_2}\, e^{\theta J_3}$, show that
$$O[\theta', a_1', a_2']\, O[\theta, a_1, a_2] = O[\theta' + \theta,\ \alpha_1(\theta') + a_1',\ \alpha_2(\theta') + a_2']\,.$$
   Deduce from this that $ISO(2)$ is isomorphic to $SO(2) \ltimes T^2$, where T^2 is the 2-dimensional translation group.

   (d) Take $|k, e_1, e_2\rangle$ to be an eigenvector of E_1 and E_2 with eigenvalues e_1 and e_2, respectively. Show that there are a continuum of eigenvalues for E_1 and E_2. Given that massless states like neutrinos do not have a continuous internal degree-of-freedom, what does that imply about physically allowed values e_1 and e_2?

5. Show that there is a choice of basis for $L(SO(4))$ consisting of 4×4 antisymmetric matrices that contain precisely two non-zero entries: 1 and -1. Evaluate the commutation relations of these generators. By choosing a new basis consisting of sums and differences of pairs of $L(SO(3))$ generators, show that $L(SO(4)) \cong L(SO(3)) \oplus L(SO(3))$.

6. Let $\{T^i{}_j\}$ be $n \times n$ matrices such that $T^i{}_j$ has a 1 in the i'th row and j'th column and is zero otherwise.

   (a) Show that they satisfy the Lie algebra
$$[T^i{}_j, T^k{}_l] = \delta^k{}_j T^i{}_l - \delta^i{}_l T^k{}_j.$$

   (b) Define $X = T^i{}_j X^j{}_i$ with arbitrary components $X^j{}_i$. Determine the adjoint matrix $(X^{\mathrm{ad}})^n{}_m{,}^k{}_l$ by
$$[X, T^k{}_l] = T^m{}_n (X^{\mathrm{ad}})^n{}_m{,}^k{}_l,$$
   and show that
$$\kappa(X, Y) = \mathrm{Tr}\big(X^{\mathrm{ad}} Y^{\mathrm{ad}}\big) = 2\big(n \textstyle\sum_{i,j} X^j{}_i Y^i{}_j - \sum_i X^i{}_i \sum_j Y^j{}_j\big).$$

(c) Show that $1 + \epsilon X \in U(n)$ for infinitesimal ϵ if $(X^j{}_i)^* = -X^i{}_j$.

(d) Hence show that in this case

$$\kappa(X,X) = -2n\textstyle\sum_{i,j}|\hat{X}^j{}_i|^2, \qquad \hat{X}^j{}_i = X^j{}_i - \frac{1}{n}\delta^j{}_i\textstyle\sum_k X^k{}_k,$$

and therefore $\kappa(X,X) = 0 \Leftrightarrow X^{\mathrm{ad}} = 0$.

(e) What restrictions must be made for $SU(n)$ and verify that in this case the generators satisfy $\kappa(X,X) < 0$ so the group is semi-simple?

7. For a simple Lie algebra $\mathfrak{g}$, with elements X_a such that $[X_a, X_b] = f_{abc}X_c$ where f_{abc} is totally antisymmetric, let $\tilde{T}_a$ be matrices forming a basis for representation R of $\mathfrak{g}$, and assume $\tilde{T}_a\tilde{T}_a = C_R I$. Define

$$\langle X_a, X_b\rangle = \mathrm{Tr}(\tilde{T}_a\tilde{T}_b)\frac{\dim\mathfrak{g}}{C_R\dim R}\,.$$

(a) Let $\mathfrak{g} = \mathfrak{su}(2)$. Evaluate $\langle J_3, J_3\rangle$ in the j-th irreducible representation of $\mathfrak{su}(2)$ and show that the result is independent of j.

(b) For $\mathfrak{su}(3)$ show that the Gell-Mann representation, $\tilde{T}_a = \frac{i}{2}\lambda_a$, where the Gell-Mann matrices λ_a are given below, gives the same value for $\langle X_a, X_b\rangle$ as does the adjoint representation $(T_a^{\mathrm{ad}})_{bc} = f_{abc}$.
[The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\quad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad \lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\quad \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\ .]$$

(c) It can be shown that the Killing form on a simple Lie algebra is the unique ad-invariant, symmetric bilinear form, up to an overall scalar multiple. How do you interpret your calculations above in relation to this fact?

8. The Lie algebra of $U(n)$ may be represented by a basis consisting first of the $n^2 - n$ off-diagonal matrices $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i \neq j$ and also the n diagonal matrices $(h_i)_{kl} = \delta_{ik}\delta_{kl}$, (no sum on k), where $i, j, k, l = 1, \ldots n$. For $SU(n)$ it is necessary to restrict to traceless matrices given by $h_i - h_j$ for some i, j. The $n-1$ independent $h_i - h_j$ correspond to the Cartan subalgebra.

(a) Show that

$$[h_i, E_{jk}] = (\delta_{ij} - \delta_{ik})E_{jk}, \qquad [E_{ij}, E_{ji}] = h_i - h_j \ \text{ (no summation convention).}$$

(b) Let $\mathbf{e}_i$ be orthogonal n-dimensional unit vectors, $(\mathbf{e}_i)_j = \delta_{ij}$. Show that E_{ij} is associated with the root vector $\mathbf{e}_i - \mathbf{e}_j$.

(c) Hence show that there are $n(n-1)$ root vectors belonging to the $n-1$ dimensional hyperplane orthogonal to $\sum_i \mathbf{e}_i$.

(d) Verify that we may take as simple roots

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \; \ldots, \quad \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \; \ldots, \quad \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,$$

by showing that all roots may be expressed in terms of the α_i with either positive or negative integer coefficients.

(e) Determine the Cartan matrix and write down the corresponding Dynkin diagram. [You may assume the Killing form is diagonal.]

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