Examples Sheet 1

1. (Warm-up) Writing the amplitude for a quantum mechanical particle, mass m, to freely propagate in 1-dimension as

$$\langle x|e^{-iH(t-t_0)}|x_0\rangle \equiv K(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i(t-t_0)}}\exp\left(\frac{im(x-x_0)^2}{2(t-t_0)}\right)$$

with $t > t_0$ (and in natural units $\hbar = 1$), show that

$$\int dx'\, K(x,t;x',t')\, K(x',t';x_0,t_0) = K(x,t;x_0,t_0)$$

with $t > t' > t_0$. Verify, given an initial condition of $(x_0,t_0) = (0,0)$, that $\lim_{t\to 0} K(x,t;0,0) = \delta(x)$ and that

$$i\frac{\partial}{\partial t}K(x,t;0,0) = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}K(x,t;0,0)\,.$$

Use these facts to express the solution of the Schrödinger equation for a free particle, $\Psi(x,t)$ in terms of an initial wavefunction $\Psi(x_0,0)$ and $K(x,t;x_0,0)$. Check this result for $\Psi(x_0,0) = e^{ikx_0}$, with k constant.

2. Consider the quantum mechanics of a particle moving in 1-dimension with Hilbert space $\mathcal{H}$. Obtain path integral expressions, in imaginary time T, for the following,

   (a) $\mathrm{Tr}_{\mathcal{H}}(\mathsf{P}e^{-TH})$, where P is the parity operator $\mathsf{P} : x \mapsto -x$, and the trace of an operator O, $\mathrm{Tr}_{\mathcal{H}}(O)$, is the sum or integral over the expectation values of O in a complete basis of states.

   (b) $\langle\psi_f|e^{-TH}|\psi_i\rangle$, where $\psi_{i,f}(x) = \langle x|\psi_{i,f}\rangle$ are arbitrary states in the Hilbert space.

   For a particle in a 1-dimensional harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$, the amplitude for particle propagation in real time t is

$$K(x,t;x_0,0) = \sqrt{\frac{m\omega}{2\pi i\sin\omega t}}\exp\left(im\omega\frac{(x^2+x_0^2)\cos\omega t - 2xx_0}{2\sin\omega t}\right)$$

   (e.g. see Osborn's notes §1.2.1). Using this amplitude for $t = -iT$, evaluate your expressions for (a) and (b) explicitly in the case that $|\psi_{i,f}\rangle$ are the ground state of the harmonic oscillator. Check that they agree with what you expect from quantum mechanics, working directly in the energy basis.

3. Consider the partition function

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4\right) \tag{1}$$

   for a zero-dimensional QFT, given $\lambda > 0$.

(a) By expanding the integral in λ, obtain the n-th order perturbative expression

$$Z_n(\lambda) = \sum_{\ell=0}^{n} \left(-\frac{\lambda}{4!}\right)^{\ell} \frac{(4\ell)!}{4^{\ell}(2\ell)!\ell!}$$

and show that, for $\ell \leq 3$, the coefficients a_ℓ of λ^ℓ in this expression are the sums of symmetry factors of the relevant loop Feynman graphs. [*At 2-loop order there is only 1 graph, at 3-loops there are 2, and at 4-loops there are 4.*]

(b) (Optional but instructive.) Using any computer package, plot $Z_n(\lambda = 0.1)$ against n to see that there is a region in n where Z_n appears to converge before blowing up as n is increased.

(c) (* Slightly beyond the scope of the course.) Show that the minimum value of $a_\ell \lambda^\ell$ occurs when $\ell \approx \frac{3}{2\lambda}$. Hence show that the Borel transform

$$\mathcal{B}Z(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a_\ell \lambda^\ell$$

converges provided $|\lambda| < \frac{3}{2}$ and that in this case

$$Z(\lambda) = \int_0^\infty dz\, e^{-z}\, \mathcal{B}Z(z\lambda)$$

so that $Z(\lambda)$ may be recovered from its Borel transform.

(d) By expanding $e^{-x^2/2}$ in the integral in (1) obtain the strong coupling expansion

$$Z(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \Gamma\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}}$$

for $Z(\lambda)$ as a series in $1/\sqrt{\lambda}$. For $\lambda = \frac{1}{10}$ how many terms does one need in order to obtain the value at which the weak coupling expansion appeared to converge?

4. Let $e^{-W(J)/\hbar} = \int d^n\phi\, e^{-(S(\phi) - J_c\phi_c)/\hbar}$, and let $\Gamma(\Phi)$ be the Legendre transform of $W(J)$. Show directly that

$$-\hbar^2 \left.\frac{\partial^3 W}{\partial J_a \partial J_b \partial J_c}\right|_{J=0} = \langle \phi_a \phi_b \phi_c \rangle^{\text{conn}}$$

and show how this can be related to the third derivative of $\Gamma(\Phi)$

$$\left.\frac{\partial^3 \Gamma}{\partial \Phi_a \partial \Phi_b \partial \Phi_c}\right|_{J=0}.$$

Comment on why the notation $\langle \phi_a \phi_b \phi_c \rangle^{\text{conn}}_{\text{1PI}} = -\frac{1}{\hbar} \left.\frac{\partial^3 \Gamma}{\partial \Phi_a \partial \Phi_b \partial \Phi_c}\right|_{J=0}$ is sometimes used.

5. In lectures we showed that

$$\int d^{2k}\theta \, e^{\frac{1}{2}A_{ab}\theta_a\theta_b} = \mathrm{Pf}(A)$$

where A is a real, invertible antisymmetric matrix and θ_a are $2k$ Grassmann variables. By writing $\theta_a = N_{ab}\theta'_b$ for some real matrix N, show that $\mathrm{Pf}(N^TAN) = \det(N)\mathrm{Pf}(A)$. Show that N may be chosen so as to put A into the form

$$N^TAN = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix},$$

and demonstrate that consequently $\mathrm{Pf}(A) = \pm\sqrt{\det A}$.

6. Consider a theory of 4 Grassmann variables θ_a, $a = 1\ldots 4$, governed by the action

$$S(\theta) = \frac{1}{2}A_{ab}\theta_a\theta_b + \frac{1}{4!}\lambda_{abcd}\theta_a\theta_b\theta_c\theta_d\,.$$

Compute the partition function of this theory (a) by directly expanding $e^{-S/\hbar}$ in the path integral and (b) by writing down the Feynman rules and drawing all possible vacuum diagrams.

7. Let M be an $N\times N$ Hermitian matrix, g be a constant, and

$$V(M) = \frac{1}{2}\,\mathrm{Tr}M^2 + \frac{g}{4}\,\mathrm{Tr}M^4\,.$$

(a) Show that $V(M)$ is invariant under $M \mapsto U^\dagger MU$, where U is a unitary matrix. Explain why this implies that $V(M)$ only depends on the eigenvalues $\{\lambda_i\}$ of M.

(b) Viewing $V(M)$ as the action for a zero-dimensional QFT, obtain an expression for the propagator $\langle M_{ij}M_{k\ell}\rangle$ correct to lowest nontrivial order in g. [Hint: The tree-level propagator may be represented using a double line, with arrows on each of the lines pointing in opposite directions. Beyond tree-level, consider only "connected" contributions.]

(c) [Do this part only if you want an extra challenge. The underlying theory behind it will not come until the end of the course.] Now let B, C, and H be further $N\times N$ matrices, each of which have zeros all along the leading diagonal. Let the elements of B and C be fermionic variables, while the entries of H are bosonic. Consider the matrix integral

$$Z(g) = \int \frac{[dM\,dB\,dC\,dH]}{(2\pi i)^{N(N-1)}}\,\exp\left[-NV(M) + i\,\mathrm{Tr}(HM) + \mathrm{Tr}(B[M,C])\right]$$

where the measure $[dM\, dB\, dC\, dH]$ indicates an integral over each entry of M and the off-diagonal entries of B, C, and H. Obtain the effective action for the eigenvalues $\{\lambda_i\}$ of M. [Do not attempt to perform the path integral over these eigenvalues.]

Please e-mail me at `M.Wingate@damtp.cam.ac.uk` with any comments, especially any errors.