Examples Sheet 3

1. Scalar QED describes the interactions of photons with a complex scalar field. In d dimensions it is defined by the action

$$S[\phi, A] = \int d^d x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi + m^2 \phi^* \phi \right]$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$.

   (a) Show that, not including counterterms, there are two distinct one-loop Feynman graphs that contribute to vacuum polarization in scalar QED. One of these diagrams leads to an integral which is independent of the external momentum. What is its role?

   (b) By considering vacuum polarization, show that when $d = 4$, the one-loop β-function for the dimensionless coupling g corresponding to the charge e is

$$\beta(g) = \frac{g^3}{48\pi^2} \qquad (\overline{\text{MS}} \text{ scheme}) .$$

   How does the theory behave at scales far below the mass of the scalar?

   (c) From your calculation, verify that the renormalized photon self-energy at one loop is transverse; i.e. that it satisfies the condition $p_\mu \Pi^{\mu\nu}(p) = 0$.

2. Denoting the photon field in momentum space by $\tilde{A}_\mu(k)$, Furry's theorem states that $\langle \tilde{A}_{\mu_1}(k_1) \cdots \tilde{A}_{\mu_n}(k_n) \rangle = 0$ when n is odd. It is a consequence of charge conjugation invariance.

   (a) In scalar QED, charge conjugation swaps ϕ and ϕ^*. How must the photon field A_μ transform if the action is to be invariant?

   (b) Prove Furry's theorem in scalar QED using the path integral formulation.

   (c) Does Furry's theorem hold for off-shell photons with $k_\mu k^\mu \neq 0$?

3. Consider the theory of a real scalar field ϕ and massless fermionic Dirac spinor ψ with action

$$S[\phi, \psi] = \int d^4 x \left[\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \bar{\psi} \partial\!\!\!/ \psi + g \phi \bar{\psi} \gamma_5 \psi + \frac{\lambda}{4!} \phi^4 \right]$$

in 4-dimensional Euclidean space. The Dirac matrices obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, are Hermitian, and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5^\dagger$.

(a) Show that the action is invariant under the global transformation

$$\phi \mapsto -\phi \qquad \psi \mapsto e^{-\frac{i}{2}\pi\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi}e^{-\frac{i}{2}\pi\gamma_5}\,.$$

Assuming that the path integral measure is also invariant under this transformation, show that the renormalization cannot generate any vertices involving odd powers of the scalar field unless they are accompanied by an odd power of $\bar{\psi}\gamma_5\psi$, as in the original action.

(b) What counterterms are necessary? Using dimensional regularization and the on-shell renormalization scheme evaluate these counterterms to one-loop accuracy, and show that the physical amplitudes are finite.

Please e-mail me at `M.Wingate@damtp.cam.ac.uk` with any comments, especially any errors.