String Theory: Example Sheet 1

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1. Consider an electron in orbit around a neutron. Assume that electromagnetic dipole effects can be neglected and only the Newtonian gravitational force is relevant. What is the radius of Bohr orbit of the ground state? What object is this comparable to? Now re-evaluate your sense of the importance of quantum gravity.

2a. Consider the 'Polyakov' action for a massive point particle moving in Minkowski space

$$S[X,e] = \frac{1}{2}\int \left(e^{-1}(\tau)\, \eta_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - e(\tau)\, m^2 \right) d\tau \tag{†}$$

where the dynamical fields are a map $X : \mathbb{R} \to \mathbb{R}^{1,d}$ and the worldline einbein e. Show that, after substituting the equation of motion for e, one recovers the form of the point-particle action that we saw in the lectures.

b. Higher-dimensional objects are called *branes*. More precisely, if the object has p spatial dimensions, it's called a p-brane. (Joke coutesy of Paul Townsend). The dynamics of a p-brane moving in Minkowski space is given by the Dirac action,

$$S = -T\int d^{p+1}\sigma \sqrt{-\det\gamma}$$

Here σ^α are coordinates on the brane worldvolume, with $\alpha = 0, \ldots, p$, while

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}$$

is the pull-back of the Minkowski metric onto the brane. Show that this is equivalent to the Polyakov-type action with dynamical worldvolume metric $g_{\alpha\beta}$,

$$S = -\frac{T}{2}\int d^{p+1}\sigma\ \sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - (p-1)\right)$$

3. The Polyakov string is invariant under Poincaré transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$$

on the target space. Show that, in conformal gauge, the Noether currents associated to these global symmetries on the worldsheet are

$$P^\alpha_\mu = T\,\partial^\alpha X_\mu \qquad \text{and} \qquad J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu\,,$$

where T is the string tension. Write down the conserved charge associated with Lorentz transformations in terms of the modes of the string and interpret the result.

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4a. Using $g^{-1}\delta g = g^{\alpha\beta}\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta} = -g^{\alpha\gamma}g^{\beta\delta}\delta g_{\gamma\delta}$, derive an expression for the energy-momentum tensor

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}}$$

from the Polyakov action for the string.

b. Now choose a metric such that the line element on the worldsheet is $-\,\mathrm{d}\tau^2 + \mathrm{d}\sigma^2$. Show that, in worldsheet light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, $T_{\alpha\beta}$ may be written as

$$T_{++} = -\frac{1}{\alpha'}\eta_{\mu\nu}\partial_+ X^\mu\,\partial_+ X^\nu\,, \qquad T_{--} = -\frac{1}{\alpha'}\eta_{\mu\nu}\partial_- X^\mu\,\partial_- X^\nu\,, \qquad T_{+-} = T_{-+} = 0\,.$$

Hence show that $\partial_- T_{++} = 0 = \partial_+ T_{--}$.

c. Find an expression for T_{++} and T_{--} in terms of the oscillator modes α_n^μ and $\bar{\alpha}_n^\mu$ at $\tau = 0$. Hence find an expression for the Virasoro modes ℓ_n, where

$$T_{--}(\sigma) = -\sum_n \ell_n\, e^{in\sigma}, \qquad T_{++}(\sigma) = -\sum_n \bar{\ell}_n\, e^{-in\sigma}\,.$$

5. Using the Poisson bracket relation $\{\alpha_m^\mu, \alpha_n^\nu\} = -im\eta^{\mu\nu}\delta_{m+n,0}$ show that $\{\ell_m, \alpha_n^\mu\} = in\alpha_{m+n}^\mu$ and hence that $\{\ell_m, \ell_n\} = -i(m-n)\ell_{m+n}$.

6a. The open string is described by the Polyakov action, but with boundary conditions at $\sigma = 0, \pi$. Show that the equations of motion require the boundary conditions to be either

$$\delta X^\mu = 0, \qquad \text{or} \qquad \partial_\sigma X^\mu = 0$$

on the boundary. (The first are called Dirichlet boundary conditions and signal the presence of D-branes. The second are called Neumann boundary conditions.)

b. Show that the boundary conditions imply only one set of independent oscillators. Impose canonical commutation relations on the modes and show that the theory has a tachyon in the ground state.

c. Show that the state $|A\rangle = A_\mu(k)\alpha_{-1}^\mu|k\rangle$ is massless (what do you have to assume about L_0 for this to be the case?). Show that $k^\mu A_\mu = 0$ and that the state $|\lambda\rangle = \lambda(k)\,k_\mu\alpha_{-1}^\mu|k\rangle$ is spurious (*i.e.* $\langle\lambda|\lambda\rangle = 0$). Give a space-time interpretation to the Fourier transform of $A_\mu(k)$.

d. Write down the mode expansion describing an open string stretched between two parallel Dp-branes. Interpret the result.

7a. Given the canonical commutation relations, one can show that the Virasoro generators L_n in the quantum theory do not satisfy the Witt algebra, but instead satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0},$$

where mathematicians call $A(m)$ the *central extension*, while physicists call it the *quantum anomaly*. Show that the $SL(2)$ sub-algebra generated by L_0, L_1, L_{-1} has no anomaly.

b. Explain why $A(-m) = -A(m)$, and show that the Jacobi identity fixes

$$A(m) = Bm^3 + Cm,$$

for some constants B and C.

c. By considering the matrix elements $\langle 0; p|[L_1, L_{-1}]|0; p\rangle$ and $\langle 0; p|[L_2, L_{-2}]|0; p\rangle$, show that

$$A(m) = \frac{D}{12} m(m^2 - 1)\,,$$

where D is the dimension of space-time. Does this mean the quantum string is only consistent in $D = 0$ dimensions? What have we missed?

8. Show that the Ricci curvature for the the conformally flat 2d Euclidean metric $g_{\alpha\beta} = e^{2\phi}\delta_{\alpha\beta}$ is given by

$$R = -2e^{-2\phi}\,\partial^2\phi$$

9. Construct the open string states at level 2 in the lightcone formalism and determine their representation under $SO(D-1)$. Construct the states at level 3. Show that they fit into a traceless symmetric-3-tensor and an anti-symmetric-2-tensor representation of $SO(D-1)$.

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