Problem Set 1

FRW: Geometry and Dynamics

Warmup Questions

(a) What is conformal time? Why is it useful?

(b) How do the energy densities in radiation (ρ_r), matter (ρ_m) and a cosmological constant (ρ_Λ) evolve with the scale factor $a(t)$?

(c) What is $a(t)$ for a flat universe dominated by radiation, matter or a cosmological constant?

(d) What is the redshift of matter-radiation equality if $\Omega_r = 9.4 \times 10^{-5}$ and $\Omega_m = 0.32$?

(e) What is H_0^{-1} in sec and in cm?

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1. De Sitter Space

(a) Show in the context of expanding FRW models that if the combination $\rho + 3P$ is always positive, then there was a Big Bang singularity in the past. [Hint: A sketch of $a(t)$ vs. t may be helpful.]

(b) Derive the metric for a positively curved FRW model ($k = +1$) with only vacuum energy ($P = -\rho$):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ell^2 \cosh^2(t/\ell) \left[\mathrm{d}\chi^2 + \sin^2\chi\, \mathrm{d}\Omega^2\right] \ .$$

Does this model have an initial Big Bang singularity?

2. Friedmann Equation

Consider a universe with pressureless matter, cosmological constant and spatial curvature.

(a) Show that the Friedmann equation can be written as the equation of motion of a particle moving in one dimension with total energy zero and potential

$$V(a) = -\frac{4\pi G}{3}\frac{\rho_{m,0}}{a} + \frac{k}{2} - \frac{\Lambda}{6}a^2 \ .$$

Sketch $V(a)$ for the following cases: *i)* $k = 0$, $\Lambda < 0$, *ii)* $k = \pm 1$, $\Lambda = 0$, and *iii)* $k = 0$, $\Lambda > 0$. Assuming that the universe "starts" with $da/dt > 0$ near $a = 0$, describe the evolution in each case. Where applicable determine the maximal value of the scale factor.

(b) Now consider the case $k > 0$, $\Lambda = 0$. Normalise the scale factor to be unity today ($a_0 \equiv 1$) to find that $k = H_0^2(\Omega_{m,0} - 1)$. Rewrite the Friedmann equation in conformal time and confirm that the following is a solution

$$a(\tau) = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}\left[1 - \cos(\sqrt{k}\tau)\right] \ .$$

Integrate to obtain the proper time

$$t(\tau) = H_0^{-1}\frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)^{3/2}}\left[\sqrt{k}\tau - \sin(\sqrt{k}\tau)\right] \ .$$

Show that the universe collapses to a 'big crunch' at $t_{\rm BC} = \pi H_0^{-1}\Omega_{m,0}(\Omega_{m,0} - 1)^{-3/2}$. How many times can a photon circle this universe before $t_{\rm BC}$?

3. Flatness Problem

Consider an FRW model dominated by a perfect fluid with pressure $P = w\rho$, for $w = const$. Define the time-dependent density parameter

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\rm crit}(t)} \ ,$$

where $\rho_{\rm crit}(t) \equiv 3H^2/8\pi G$. Show that

$$\frac{d\Omega}{d\ln a} = (1 + 3w)\Omega(\Omega - 1) \ .$$

Discuss the evolution of $\Omega(a)$ for different initial conditions and different values of w.

4. Einstein's Biggest Blunder

(a) Show that for a physically reasonable perfect fluid (i.e. density > 0 and pressure ≥ 0) there is no static isotropic homogeneous solution to Einstein's equations.

(b) Show that it is possible to obtain a static zero-pressure solution by the introduction of a cosmological constant Λ such that

$$\Lambda = 4\pi G\rho_{m,0} \ .$$

However, show that this solution is unstable to small perturbations.

5. Accelerating Universe

Consider flat FRW models ($k = 0$) with pressureless matter ($P = 0$) and a non-zero cosmological constant $\Lambda \neq 0$, that is, with $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$.

(a) *Age of the universe.*—Show that the normalised solution ($a_0 \equiv 1$) for $\Omega_{m,0} \neq 0$ can be written as

$$a(t) = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \left(\sinh\left[\tfrac{3}{2}H_0(1-\Omega_{m,0})^{1/2}t\right]\right)^{2/3} .$$

Verify that $a(t)$ has the expected limits at early times, $H_0 t \ll 1$, and at late times, $H_0 t \gg 1$. Hence show that the age of the universe t_0 in these models is

$$t_0 = \tfrac{2}{3}H_0^{-1}(1-\Omega_{m,0})^{-1/2}\sinh^{-1}\left[(\Omega_{m,0}^{-1}-1)^{1/2}\right] ,$$

and roughly sketch this as a function of $\Omega_{m,0}$.

(b) Λ-*domination and acceleration.*—Show that the energy density of the universe becomes dominated by the cosmological constant term at the following redshift

$$1 + z_\Lambda = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3} ,$$

but that it begins accelerating earlier at $1 + z_A = 2^{\frac{1}{3}}(1 + z_\Lambda)$.

(c) *Causal structure and future.*—Show that the furthest object with which we can communicate is today at a physical distance

$$\int_0^1 \frac{H_0^{-1}\,\mathrm{d}x}{\sqrt{1-\Omega_{m,0}+\Omega_{m,0}x^3}} .$$

Argue that this implies the existence of a future event horizon (for all $\Omega_{m,0} < 1$). By integrating back in time, show that the redshift $1 + z_{\rm eh}$ of these objects can be found by equating

$$\int_1^{1+z_{\rm eh}} \frac{\mathrm{d}x}{\sqrt{1-\Omega_{m,0}+\Omega_{m,0}x^3}} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-\Omega_{m,0}+\Omega_{m,0}x^3}} .$$

For the 'concordance' model ($\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$) we find $z_{\rm eh} \approx 1.8$ and so the many galaxies and quasars observed beyond this will be forever inaccessible. What caveats might affect this conclusion?