Problem Set 3

Structure Formation

Warmup Questions

(a) How do density perturbations in a pressureless fluid grow (i) in a static space and (ii) in an expanding space?

(b) Explain the Mészáros effect.

(c) Explain the assumptions that go into the Press-Schechter mass function

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1. Newtonian Structure Formation

In class we derive evolution equations for the perturbations from GR but on subhorizon scales we can use a Newtonian treatment. Consider the equations of fluid dynamics in the presence of a gravitational potential:

$$\begin{aligned} &\text{continuity} \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\rho \boldsymbol{u}) = 0 \ , \\ &\text{Euler} \qquad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}) \boldsymbol{u} + \frac{1}{\rho} \boldsymbol{\nabla}_{\boldsymbol{r}} P + \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi = 0 \ , \\ &\text{Poisson} \qquad \nabla^2_{\boldsymbol{r}} \Phi = 4\pi G \rho \ . \end{aligned}$$

(a) Show that the homogeneous expanding universe corresponds to the following solution

$$\bar{\rho}(t) = \frac{\bar{\rho}(t_0)}{a^3(t)} \ , \qquad \bar{\boldsymbol{u}} = \frac{\dot{a}}{a} \boldsymbol{r} \ , \qquad \boldsymbol{\nabla}_{\boldsymbol{r}} \bar{\Phi} = \frac{4\pi G}{3} \bar{\rho} \boldsymbol{r} \ .$$

(b) Consider linear perturbations about the homogeneous solution. Show that the density perturbations satisfy

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + \left[\frac{c_s^2 k^2}{a^2} - 4\pi G \bar{\rho}\right] \delta = 0 \ , \qquad (\star)$$

where $\delta \equiv \delta\rho/\bar{\rho}$ and $c_s^2 \equiv \delta P/\delta\rho$.

(c) How does $(\star)$ have to be modified in order to describe the evolution of dark matter perturbations in our universe? Find the solutions to this new equation (i) during radiation domination, (ii) during matter domination, and (iii) during dark energy domination.

[Hint: First explain why you may ignore perturbations in the radiation and the dark energy.]

2. Growth of Matter Perturbations I: Early Times

At early times, the universe was dominated by radiation (r) and pressureless matter (m). You may ignore baryons.

(a) Show that the conformal Hubble parameter satisfies

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r} \left(\frac{1}{y} + \frac{1}{y^2} \right) ,$$

where $y \equiv a/a_{\rm eq}$ is the ratio of the scale factor to its value when the energy density of the matter and radiation are equal.

(b) Describe qualitatively the behaviour of the Newtonian-gauge fractional density perturbations δ_r and δ_m for a scalar perturbation at scale k, with adiabatic initial conditions, that re-enters the Hubble radius well before $a_{\rm eq}$.

(c) For perturbations on scales much smaller than the Hubble radius, the fluctuations in the radiation can be neglected. Assuming that Φ evolves on a Hubble timescale, show that

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \bar{\rho}_m \delta_m \approx 0 . \qquad (\star)$$

(d) Show that, in terms of the variable y, eq. ($\star$) becomes

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)} \frac{d\delta_m}{dy} - \frac{3}{2y(1+y)} \delta_m = 0 .$$

Hence verify that the solutions are

$$\delta_m \propto \begin{cases} 2+3y \\ (2+3y) \ln\left(\dfrac{\sqrt{1+y}+1}{\sqrt{1+y}-1} \right) - 6\sqrt{1+y} \end{cases}$$

Determine how δ_m grows with y for $y \ll 1$ (RD) and $y \gg 1$ (MD).

3. Growth of Matter Perturbations II: Late Times

At late times, the universe is dominated by pressureless matter (m) and dark energy (Λ). Assuming that the dark energy doesn't cluster, the gravitational potential is only sourced by the matter.

(a) Use the Einstein equations to show that the comoving-gauge matter density contrast, $\Delta_m \equiv \delta_m - 3\mathcal{H}v_m$, evolves as

$$\Delta_m'' + \mathcal{H}\Delta_m' - 4\pi G a^2 \bar{\rho}_m \Delta_m = 0 .$$

(b) Show that $u \equiv \Delta_m / H$ satisfies

$$\frac{d^2u}{da^2} + 3\frac{d\ln(aH)}{da}\frac{du}{da} = 0 \; .$$

Confirm that the decaying mode is $\Delta_m \propto H$, while the growing mode can be written as

$$\Delta_m \propto H \int_a^{a_i} \frac{\mathrm{d}\tilde{a}}{(\tilde{a}\tilde{H})^3} \; .$$

(c) What are the growing and decaying modes of Δ_m in the matter-dominated era? What is the asymptotic limit ($a \to +\infty$) of the growing mode solution in the dark energy-dominated era?

4. Matter Power Spectrum In class we used the evolution of Δ_m to calculate the matter power spectrum we observe in terms of the power spectrum of ζ. Reproduce the argument but using δ_m instead of Δ_m to obtain the same result.

5. Spherical Collapse of Radiation Suppose we wanted to get an idea of how radiation density perturbations evolved during the radiation era. We could try to naively repeat our arguments regarding spherical collapse of dark matter but with radiation. If we make a overdensity of radiation by compressing a sphere from $\overline{R}$ to R then we can make the same argument that the two regions will be gravitationally decoupled but the pressure discontinuity on the boundary will stop the regions from fully decoupling. However if we consider perturbations which are much larger than the horizon the information about the pressure discontinuity can't travel very far into the interior and the effect can be neglected (this is the separate universe approach which we can use for superhorizon modes. Where we expect the modes to evolve purely by their modification of the the local background). Assuming this is the case we can decouple the two regions. Now the outside is described by;

$$\left(\frac{\dot{\overline{a}}}{\overline{a}}\right)^2 = \frac{H_0^2\overline{\Omega}_{r,0}}{\overline{a}^4} = \frac{H_0^2}{\overline{a}^4}$$

and the inside by;

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2\Omega_{r,0}}{a^4} - \frac{K}{a^2}$$

Show that these have solutions

$$\begin{aligned} \overline{a} &= \sqrt{2H_0 t} \\ R &= A\sin(\theta) \\ t &= B\,(1-\cos\theta) \end{aligned}$$

where you should determine A, B and θ. Now expand the solution for a in terms of θ to show that at zeroth order

$$R = \overline{R}$$

as we expect (you will need that $\Omega_{r,0}^{1/4} = \overline{R}_0/R_0$ as the density of radiation goes as a^4 rather than a^3 as for matter). Show that at first order the linear density perturbation grow as

$$\delta_{linear} = \frac{\Omega_{r,0} - 1}{2\sqrt{\Omega_{r,0}}}\overline{a}^2 \propto t$$

So we have recovered the result from the previous section that on superhorizon scales $\Delta_r \propto \tau^2 \propto a^2$.

6. Virial Theorem

We used the Virial theorem for non-relativistic gravitating bodies to determine the radius at which spherical over densities stop collapsing. Here we will prove it. First take the rotational inertia of the system

$$I = \sum_i m_i \boldsymbol{r}_i \cdot \boldsymbol{r}_i$$

where i runs over particles in the system. We define the virial, G, as

$$G \equiv \frac{1}{2}\frac{dI}{dt}$$

Show

$$\frac{dG}{dt} = 2K + \sum_i \mathbf{F}_i \cdot \boldsymbol{r}_i$$

where $\mathbf{F}_i$ is the force on particle i and K is the total kinetic energy. Now use $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$, where $\mathbf{F}_{ij}$ is the force exerted on particle i by particle j, and that the gravitational force between particles will be the gradient of the gravitational potential, $\mathbf{F}_{ij} = -\nabla V_{ij}$, where

$$V_{ij} = -\frac{Gm_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

to show that

$$\sum_i \mathbf{F}_i \cdot \boldsymbol{r}_i = \sum_i \sum_{j<i} V_{ij} \equiv V$$

where V is the total potential energy. Now we have that

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + V$$

For any stable system we expect $\frac{d^2I}{dt^2} = 0$ (at least when averaged over some finite time scale) which gives us the virial theorem (for non-relativistic gravitating systems)

7. Halo Formation

One consequence of the virial theorem is that we can estimate the time at which halos formed from their mass M and their velocity dispersion σ_v. We know from our discussion of spherical collapse that

$$\rho_{vir} \approx 200\bar{\rho}(t_{vir})\,. \tag{*}$$

Write down expressions for the kinetic energy and potential energy of the halo and use the virial theorem to show,

$$R_{vir} = \frac{GM}{\sigma_v^2}\,.$$

Now write down the background density at the time of virialisation, $\bar{\rho}(t_{vir})$, in terms of $\Omega_{m,0}$ and z_{vir} and the halo density ρ_{vir} in terms of M and R_{vir}. Then use (*) to show,

$$1 + z_{vir} \leq \left(100G^2H_0^2\Omega_{m,0}\right)^{-\frac{1}{3}}\left(\frac{\sigma_v}{M^{\frac{1}{3}}}\right)^2\,.$$

This tells us that low-mass high-velocity objects formed first and high-mass low-velocity objects formed last. For our galaxy we have $\sigma_v \approx 300kms^{-1}$ and $M \approx 10^{12}M_{\text{sun}}$ we find that $z_{vir} \leq 7$. In general it is hard to form anything before $z = 10$ which defines the cosmological dark ages.