Part III Applications of Differential Geometry to Physics, Sheet One

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1. Show, by exhibiting the coordinate charts, that the real projective space $\mathbb{RP}^n$ is a manifold. Show that $\mathbb{RP}^n$ may be regarded as the n–sphere S^n with antipodal points identified. Prove that $\mathbb{RP}^3 \equiv SO(3)$. Show also that $\mathbb{RP}^n \equiv S^n/\mathbb{Z}_2$ and that $S^n \equiv O(n+1)/O(n)$.

   The complex projective space $\mathbb{CP}^n$ is defined analogously to $\mathbb{RP}^n$, as a set of one–dimensional *complex* subspaces in $\mathbb{C}^{n+1}$. Prove that, as real manifolds, $\mathbb{CP}^1 \equiv S^2$.

   Remark. Thus S^2 has an atlas with holomorphic transition functions which makes it a *complex manifold*. It is known that no other sphere apart from S^6 is a complex manifold. It is still not known whether S^6 is a complex manifold.

2. Show that the Lie algebra of $SO(n) = \{A \in GL(n, \mathbb{R}), A^T A = \mathbf{1}\}$ may be identified with antisymmetric $n \times n$ matrices.

   Let J be a $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

   and let $Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}), A^T J A = J\}$. Compute the dimensions of $SO(n)$ and $Sp(2n, \mathbb{R})$. What is the Lie algebra of $Sp(2n, \mathbb{R})$?

3. Starting from the definition of the Lie derivative show that

$$\mathcal{L}_V(W) = [V, W]$$

   if V and W are vector fields. Use the Leibniz rule to establish the Cartan formula

$$\mathcal{L}_V \Omega = d(V \lrcorner \Omega) + V \lrcorner d\Omega,$$

   where Ω is a p–form.

   Show that, if Ω is a one–form, then

$$d\Omega(V, W) = V(W \lrcorner \Omega) - W(V \lrcorner \Omega) - [V, W] \lrcorner \Omega.$$

4. Consider the matrix representation of the Euclidean group E(2) in two dimensions
$$\begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$
to find a basis of right- and left-invariant one forms and the dual vector fields. How is this matrix representation related to the action of E(2) on $\mathbb{R}^2$ discussed in Lectures?

The location of a motor car with rear wheel drive may be specified by giving the coordinates (a, b) of the centre of the front axle and the angle θ that the axis of the car makes with the a-axis. Show that the configuration space of of the car may be regarded as E(2). If l is the distance between the mid-points of the rear and front axles, show that the vector field $\mathbf{V}_\psi$ associated with driving forward the front wheels making a constant angle $\frac{\pi}{2} - \psi$ to the axis of the car is given by

$$\mathbf{V}_\psi = \cos\psi \cos\theta \frac{\partial}{\partial a} - \cos\psi \sin\theta \frac{\partial}{\partial b} + \sin\psi \frac{1}{l} \frac{\partial}{\partial \theta}$$

Show that a basis for $\mathfrak{e}(2)$ is given by $\mathbf{Steer} = \mathbf{V}_{\frac{\pi}{2}}$, $\mathbf{Drive} = \mathbf{V}_0$, and $\mathbf{Left} = [\mathbf{Steer}, \mathbf{Drive}]$. Calculate the commutation relations. Show in particular how, in the UK, parking may be achieved by a succession of infinitesimal steering and driving.

5. Consider three one–parameter groups of transformations of $\mathbb{R}$

$$x \to x + \varepsilon_1, \quad x \to e^{\varepsilon_2} x, \quad x \to \frac{x}{1 - \varepsilon_3 x},$$

and find the vector fields V_1, V_2, V_3 generating these groups. Deduce that these vector fields generate a three-parameter group of transformations
$$x \to \frac{ax + b}{cx + d}, \qquad ad - bc \neq 0.$$
Show that the vector fields V_α generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and thus deduce that $\mathfrak{sl}(2, \mathbb{R})$ is a subalgebra of the infinite dimensional Lie algebra $\mathrm{vect}(\Sigma)$ of vector fields on $\Sigma = \mathbb{R}$. Find all other finite dimensional subalgebras of $\mathrm{vect}(\Sigma)$.

Let $f : \Sigma \to \mathbb{R}$ be a smooth function. Consider a map $\pi : C^\infty(\Sigma) \to C^\infty(T^*\Sigma)$ given by

$$\pi(f)(x,p) = pf(x), \quad \text{where } (x,p) \in T^*\Sigma$$

and show that this map gives a homomorphism between $\mathrm{vect}(\Sigma)$ and the Lie algebra of Poisson bracket on $T^*\Sigma$.

Remark. The Poisson bracket on $T^*\Sigma$ admits a deformation to the so called *Moyal bracket* (if you want to, look it up on Wikipedia) which makes quantisation possible. On the other hand the algebra $\mathrm{vect}(\Sigma)$ can be centrally extended to the Virasoro algebra as discussed in lectures, but is otherwise rigid.

6. The dilatations $\mathbb{R}_+$ and translations $\mathbb{R}^4$ combine as the semi-direct product $\mathbb{R}_+ \ltimes \mathbb{R}^4$ to act on $y^\mu \in \mathbb{E}^{3,1}$, the Minkowski space-time, as

$$\begin{pmatrix} y^\mu \\ 1 \end{pmatrix} \to \begin{pmatrix} \lambda & x^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^\mu \\ 1 \end{pmatrix} . \tag{1}$$

Show that

$$ds^2 = \frac{1}{\lambda^2}\Big(d\lambda^2 + \eta_{\mu\nu} dx^\mu dx^\nu\Big)$$

is a left-invariant metric on $\mathbb{R}_+ \ltimes \mathbb{R}^4$. By considering the embedding into $\mathbb{E}^{4,2}$ given by

$$X^6 + X^5 = \frac{1}{\lambda}\,, \quad X^6 - X^5 = \lambda + \frac{\eta_{\mu\nu}x^\mu x^\nu}{\lambda}\,, \quad X^\mu = \frac{x^\mu}{\lambda}\,,$$

with X^6 an extra timelike coordinate and X^5 an extra spacelike coordinate, show that $\mathbb{R}_+ \ltimes \mathbb{R}^4$ with this metric is one half of five-dimensional Anti-de-Sitter space-time AdS_5.

Show that, despite being a group manifold , $\mathbb{R}_+ \ltimes \mathbb{R}^4$ equipped with this metric is geodesically incomplete.

Remark. This construction is currently quite popular because it is the basis of the *AdS/CFT correspondence*.

7. Let $A \in SO(3)$. Find the vector fields generating the action $\mathbf{x} \to A\mathbf{x}$ of $SO(3)$ on $\mathbb{R}^3$. Show that this action restricts to $S^2 \subset \mathbb{R}^3$, and that the symplectic form $d(\cos\theta) \wedge d\psi$ on S^2, where (θ, ψ) are spherical polars,

is preserved by the action. Deduce that the action on the two–sphere is generated by Hamiltonian vector fields, and find the corresponding Hamiltonians. Verify that these Hamiltonians form a Lie algebra with a Poisson bracket, which is isomorphic to the Lie algebra of $SO(3)$.

8. A Poisson on structure on $\mathbb{R}^{2n}$ is an anti–symmetric matrix ω^{ab} whose components depend on the coordinates $x^a \in \mathbb{R}^{2n}, a = 1, \cdots, 2n$ and such that the Poisson bracket

$$\{f, g\} = \sum_{a,b=1}^{2n} \omega^{ab}(x) \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^b}$$

satisfies the Jacobi identity.

Show that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

Assume that the matrix ω is invertible with $W := (\omega^{-1})$ and show that the antisymmetric matrix $W_{ab}(\xi)$ satisfies

$$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0, \qquad (2)$$

or equivalently that the two–form $W = (1/2)W_{ab}dx^a \wedge dx^b$ is closed.

[Hint: note that $\omega^{ab} = \{x^a, x^b\}$.] Deduce that if $n = 1$ then any anti-symmetric invertible matrix $\omega(x^1, x^2)$ gives rise to a Poisson structure (i.e. show that the Jacobi identity holds automatically in this case).

Remark. The invertible antisymmetric matrix W which satisfies (2) is called a symplectic structure. We have therefore deduced that symplectic structures are special cases of Poisson structures.

9. The metric of hyperbolic 3-space H^3 in Beltrami coordinates is given by

$$ds^2 = \frac{d\mathbf{r}^2}{1 - r^2} + \frac{(\mathbf{r}.d\mathbf{r})^2}{(1 - r^2)^2}\,.$$

Let

$$\mathbf{M} = \mathbf{p} - \mathbf{r}(\mathbf{p}.\mathbf{r})\,, \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}\,,$$

so that $\mathbf{M}.\mathbf{L} = 0$. Show that the Hamiltonian for geodesic motion is given by

$$H = \frac{1}{2}\Big(\mathbf{M}^2 - \mathbf{L}^2\Big)\,.$$

Obtain the Poisson brackets

$$\{L_i, L_j\} = \epsilon_{ijk} L_k\,,$$
$$\{M_i, M_j\} = -\epsilon_{ijk} L_k\,,$$
$$\{L_i, M_j\} = \epsilon_{ijk} M_k\,.$$

Hence show that both $\mathbf{L}$ and $\mathbf{M}$ are constants of the motion. Identify the associated Killing vector fields and compute their Lie brackets. Show that

$$\mathbf{M} = \frac{\dot{\mathbf{r}}}{1 - r^2}$$

and hence that the geodesics are straight lines in Beltrami coordinates. What is the geometrical significance of the condition $\mathbf{M}.\mathbf{L} = 0$?

Check that the Poisson algebra of $L_{ij} = \epsilon_{ijk} L_k$ and L_{0i} is that of the Lorentz Lie algebra $\mathfrak{so}(3,1)$, and show that H and $\mathbf{M}.\mathbf{L}$ are quadratic Casimirs.

10. The set of oriented lines in Euclidean space $\mathbb{R}^{n+1}$ may be parametrized in terms of their unit tangent vector $\mathbf{t}$ and the vector $\mathbf{p}$ joining the an arbitrary origin 0 to the point P of nearest approach of the line to this origin. Identify the space of oriented lines as TS^n - the tangent bundle to the n–dimensional sphere.

    Now Consider $n = 2$.

    (a) Show that points $P \in \mathbb{R}^3$ corresponds to maps L_P from S^2 to TS^2 which should be constructed. Let $\tau : TS^2 \to TS^2$ be a fixed-point-free map obtained by reversing the orientation of each straight line. Show that a two–sphere in TS^2 corresponding to $P \in \mathbb{R}^3$ is preserved by τ.

    (b) Describe the action and orbits of rotations about O on TS^2. How does the Euclidean group $E(3)$ act? What happens if we consider *unoriented* lines?

Remark. You have established a *mini–twistor correspondence* between points in $\mathbb{R}^3$ and spheres in TS^2. If a complex atlas (see Question 1) is used on S^2, then TS^2 becomes a complex manifold, and holomorphic functions on this manifold give rise to solutions of linear and non–linear PDEs on $\mathbb{R}^3$ (like the Bogomolny equations for magnetic monopoles).