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Part III Applications of Differential Geometry to Physics, Sheet Two

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1. Let $\phi : \mathbb{R}^{2,1} \rightarrow S^2$. Set

$$\phi^1 + i\phi^2 = \frac{2u}{1+|u|^2}, \qquad \phi^3 = \frac{1-|u|^2}{1+|u|^2},$$

and deduce that the Bogomolny equations

$$\partial_i \phi^a = \pm \varepsilon_{ij} \varepsilon^{abc} \phi^b \partial_j \phi^c, \qquad \phi_t = 0$$

imply that u is holomorphic or antiholomorphic in $z = x_1 + ix_2$.

Find the expression for the total energy

$$E[\phi] = \frac{1}{2} \int \partial_j \phi^a \partial_j \phi^a d^2x$$

in terms of u.

By counting the pre–images or otherwise find the topological degree of ϕ corresponding to $u(z) = u_0 + u_1 z + \ldots + u_k z^k$, where $u_0, \ldots, u_k$ are constants with $u_k \neq 0$.

2. Derive the $SU(2)$ Yang–Mills theory on $\mathbb{R}^4$ form the action. Let $A_a(x)$ be a solution to these equations. Show that, for any nonzero constant c, the potential $\widetilde{A}_a(x) = cA_a(cx)$ is also a solution and that it has the same action.

3. Consider the map $g : S^3 \rightarrow SU(2)$ defined by

$$g(x_1, x_2, x_3, x_4) = x_4 + i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3),$$

where σ_i are Pauli matrices and $x_1^2 + x_2^2 + x_3^3 + x_4^2 = 1$ and find its degree. By calculating $Tr((dg\, g^{-1})^3)$ at the point on S^3 where $x_4 = 1$, or otherwise deduce that the formula

$$\deg(g) = \frac{1}{24\pi^2} \int_{S^3} Tr((dg\, g^{-1})^3)$$

is correctly normalised.

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4. Let T_1, T_2, T_3 form a basis of $\mathfrak{su}(2)$ such that

$$[T_\alpha, T_\beta] = -\varepsilon_{\alpha\beta\gamma} T_\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3,$$

and let the symbols $\sigma_{ab} = -\sigma_{ba}$ where $a, b = 1, \ldots, 4$ be defined by

$$\sigma_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} T_\gamma, \quad \sigma_{\alpha 4} = T_\alpha.$$

Show that

$$\sigma_{ab} = \frac{1}{2} \varepsilon_{ab}{}^{cd} \sigma_{cd}, \quad \text{and} \quad \sigma_{ab}\sigma_{ac} = -\frac{3}{4} \mathbf{1} \delta_{bc} - \sigma_{bc}.$$

Identify $\Lambda^2(\mathbb{R}^4)$ with the Lie algebra $\mathfrak{so}(4)$ and deduce that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

5. Let $V = 1 + r^{-2}$, where $r^2 := \delta_{ab} x^a x^b$. Show that the one–form

$$A = \sigma_{ab} \, \frac{1}{V} \frac{\partial V}{\partial x^b} dx^a \qquad (1)$$

is a solution of the anti–self–dual Yang–Mills equations on $\mathbb{R}^4$.

The one–form A is singular at $r = 0$. What can you say about the behaviour of the field strength F at $r = 0$?

6. Find, by explicit integration, the Chern number of the solution (1).

7. Let F be a two–form on $\mathbb{R}^4$. Show, from the definition of the Hodge operator, that

   (a) $**F = \pm F$ depending on the signature.

   (b) $*F \wedge *F = F \wedge F$.

   Show that in the $U(1)$ theory $F \to *F$ interchanges the electric and magnetic fields with factors of ± 1 or $\pm i$ and determine the different cases in the corresponding signatures.

   Let F be a non–zero real self–dual two–form on $\mathbb{R}^4$ such that $F \wedge F = 0$. What is the signature of the underlying metric?

8. Let A be a 1–form gauge potential on $\mathbb{R}^n$ with values in $\mathfrak{su}(2)$, and let F be its curvature. Verify that $Tr(A), Tr(A \wedge A), Tr(A \wedge A \wedge A \wedge A)$ and $Tr(F)$ all vanish.

   Verify that $C_2 = dY_3$, where C_2 and Y_3 are the second Chern form, and the Chern–Simons three–form respectively.

9. Let $A = A_i dx^i, i = 1, 2, 3$ be a gauge potential on $\mathbb{R}^3$ with values in the Lie algebra $\mathfrak{g}$. Find the Euler–Lagrange equations arising from varying the Chern–Simons functional

$$W[A] = \int_{\mathbb{R}^3} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

with respect to A.

Now consider a one parameter family of $\mathfrak{g}$–valued one–forms $A = A(t)$ on $\mathbb{R}^3$, and define a one–form on $\mathbb{R}^4$ by $\mathcal{A} = A + \phi dt$, where the function $\phi = \phi(x^i, t)$ takes its values in $\mathfrak{su}(2)$. Show that, in a gauge where $\phi = 0$, the anti–self–dual Yang–Mills equations on $\mathcal{A}$ take the gradient flow form

$$\frac{dA_i(t)}{dt} = \frac{\delta W[A]}{\delta A_i}.$$

10. Consider a connection $\omega = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma$ on a principal G–bundle $P \to B$, where A is a one–form on B and $\gamma^{-1} d\gamma$ is the Maurer–Cartan form on G.

   (a) Show that the transformation of the fibres $\gamma' = g\gamma$, where $g \in G$ depends on the coordinates on B, does not change ω if A transforms like a gauge potential.

   (b) Let $\Omega = d\omega + \omega \wedge \omega$. Show that $\Omega = \gamma^{-1} F \gamma$ for some F which should be found.

   (c) Let $D_a, a = 1, \ldots, \dim(B)$ be linearly independent vector fields on P such that

   $$D_a \lrcorner\, \omega = 0.$$

   Show that $D_a = \partial_a - A_a^\alpha R_\alpha$, where $\partial_a = \partial/\partial x^a$ are vector fields on B and R_α are right–invariant vector fields on G. Demonstrate that

   $$[D_a, D_b] = -F_{ab}^\alpha R_\alpha.$$

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